Recap

- Last class (January 23, 2007)
  - Extensive form of a game
  - Information sets
  - Subgames
  - Subgame perfect Nash equilibrium
  - Credible Commitment
- Today (January 25, 2007)
  - Finitely repeated games
  - Infinitely repeated games

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Infinitely Repeated Prisoner's Dilemma

Prisoner 2

Prisoner 1	C (cooperate)	D (defect)
C	4, 4	0, 5
D	5, 0	1, 1

- The game is repeated infinitely
- For each t, the outcomes of the previous t-1 stage games are observed
- Payoffs?

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Discounted payoffs

- Let δ be the value today of a dollar to be received one stage later
  - E.g., $\delta=1/(1+r)$ where r is the interest rate per stage
- Given the discount factor δ, the present value of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, \ldots$ is

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \ldots = \sum_{t=1\to\infty} \delta^{t-1}\pi_t .$$

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Discounted payoffs

- Suppose after each stage is played, the game continues to the next stage with probability 1-p and stops with probability p.
- Expected present value of next stage's payoff

  $(1-p)\pi/(1+r)$.
- Expected present value of the payoff two stages later

  $(1-p)^2\pi/(1+r)^2$.
- Let $\delta= (1-p)/(1+r)$
- $\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \ldots$ reflects the time value of money and the possibility that the game will end

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Average payoffs

- $V^T = \pi_1 + \delta\pi_2 + \delta^2\pi_3 + \ldots = \sum_{t=1\to\infty} \delta^{t-1}\pi_t$
- If we received an "average" payoff of π in every stage, then
  $V^A = \pi + \delta\pi + \delta^2\pi + \ldots = \pi(1 + \delta + \delta^2 + \ldots) = \pi/(1-\delta)$
- $\pi/(1-\delta) = \sum_{t=1\to\infty} \delta^{t-1}\pi_t$.

$$\pi = (1-\delta) \sum_{t=1\to\infty} \delta^{t-1}\pi_t .$$

Example: Payoffs 4 4 4 4 4 ….
Average payoff = 4   Net present value = 4/ (1- δ)

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Infinitely repeated games

- Given a stage game G, let G(∞,d) denote the *infinitely repeated game* in which G is repeated forever and the players share discount factor d. For each t, the outcomes of the t-1 preceding plays of the stage game are observed before the t^{th} stage begins. Each player's payoff in G(∞,d) is the present value of the player's payoffs from the infinite sequence of stage games.

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Infinitely Repeated Prisoner's Dilemma

Prisoner 2

Prisoner 1

	C (cooperate)	D (defect)
C	4, 4	0, 5
D	5, 0	1, 1

- What strategies do you suggest to encourage cooperation?

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Infinitely Repeated Prisoner's Dilemma

Prisoner 2

Prisoner 1

	C (cooperate)	D (defect)
C	4, 4	0, 5
D	5, 0	1, 1

- Strategy:

  Play C in the first stage. In the t^{th} stage, if the outcome of all t-1 preceding stages has been (C,C), then play C; otherwise, play D

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Definitions

- In an infinitely repeated game $G(\infty,\delta)$, a player's *strategy* specifies the player's actions in each stage, for each possible history of play through the previous stages.

- In the infinitely repeated game $G(\infty,\delta)$, each *subgame* beginning at stage t+1 is identical to the original game $G(\infty,\delta)$.

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Trigger strategies for Prisoner's Dilemma

- Assuming player 1 adopts the trigger strategy, what is the best response of player 2?

Player 2 best response in stage t+1:

- If the outcome in stage t is (D,D)
  - Play D forever
- If the outcomes of stages 1,…,t are (C,C)
  - Play D → receive 5 in this stage, switch to (D,D) forever after → $V = 5 + \delta(1) + \delta^2(1) + \delta^3(1) + \ldots$
    $= 4 + 1/(1-\delta) = 5 + \delta/(1-\delta)$
  - Play C → receive 4 in this stage, and face the exact same game (same choices) in stage t+2!
    $V = 4 + \delta V$

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Trigger strategies for Prisoner's Dilemma

- Let V be the payoff of player 2 from making the optimal choice in the subgame starting in stage t+1, given that the outcomes in the previous stages have been (C,C)
- Play C $\rightarrow$ $V=4+\delta V \rightarrow V = 4/(1-\delta)$
- Play D $\rightarrow$ $V= 5+\delta/(1-\delta)$

Play C if $4/(1-\delta) \geq 5+\delta/(1-\delta) \rightarrow$ if $\delta \geq 1/4$

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Outcome

- Is the outcome that we found subgame perfect?
- Recall:
  - In the infinitely repeated game $G(\infty,\delta)$, each *subgame* beginning at stage t+1 is identical to the original game $G(\infty,\delta)$.
  - In each subgame we know the complete history of the game so far

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Trigger strategies for Prisoner's Dilemma

- Two types of subgames:

(i) Subgames where the outcomes of all previous stages have been (C,C)

If players adopt trigger strategies for the whole game, then strategy is again trigger for this subgame; the strategy is a Nash Equilibrium for the subgame (or the whole game).

(ii) Subgames where the outcome of at least one earlier stage differs from (C,C)

Player's strategies are to repeat (D,D) forever, which is also a Nash equilibrium for the original game

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Observation

- Even if the stage game G has a unique Nash equilibrium, there may be subgame-perfect outcomes of the infinitely repeated game in which no stage's outcome is a Nash equilibrium of G.

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Repeated Game Payoffs

- In general what kind of payoffs can be achieved in a repeated game?
- Do the average payoffs correspond to one of the pure strategy outcomes?

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Feasible payoffs in the stage game

- The payoffs $(\pi^1, \pi^2, \ldots, \pi^n)$ are ***feasible*** in the stage game G if they are a convex combination of the pure-strategy payoffs of G.

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Example

		Prisoner 2	
Prisoner 1		C (cooperate)	D (defect)
	C	4, 4	0, 5
	D	5, 0	1, 1

- What are the pure-strategy payoffs?
  - (4,4) (0,5) (5,0) (1,1)

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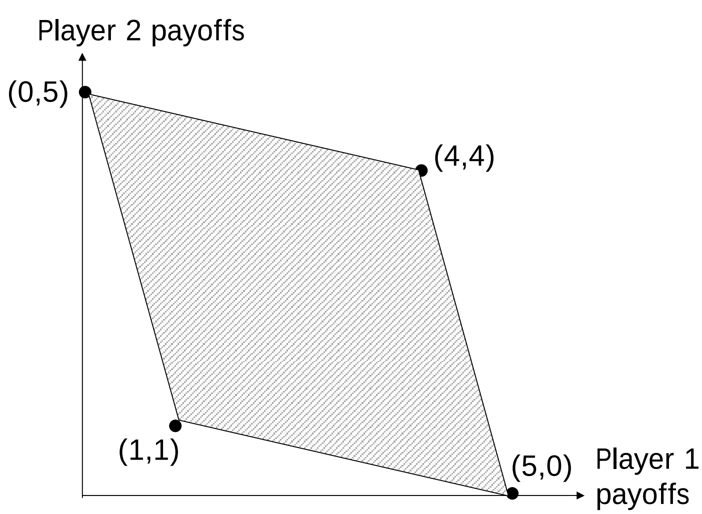
Feasible payoffs in the Prisoner's Dilemma



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Friedman's Theorem

- Let G be a finite static game of complete information. Let $(e^1, e^2, \ldots, e^n)$ denote the payoffs from a Nash equilibrium of G and let $(x^1, x^2, \ldots, x^n)$ denote any other feasible payoffs from G. If $x^j > e^j$ for every player j and if δ is sufficiently close to 1, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty,\delta)$ that achieves $(x^1, x^2, \ldots, x^n)$ as the average payoff.

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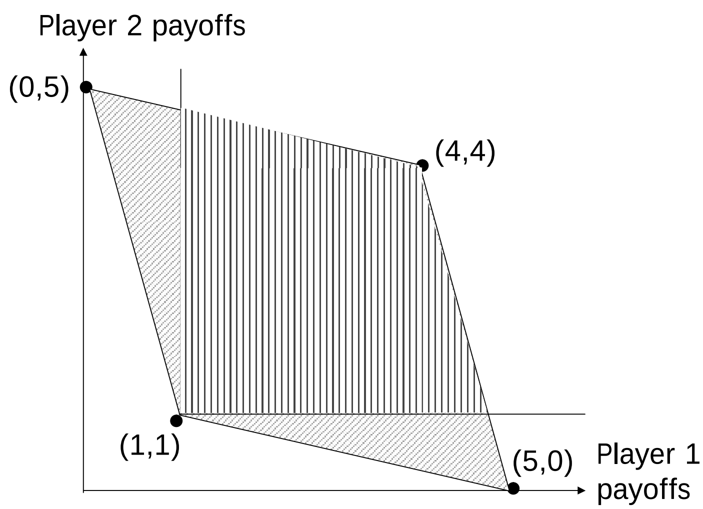
Feasible payoffs in the Prisoner's Dilemma



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