Recap:

- Last class (January 16, 2007)
  - Pareto dominance
  - Examples of games with continuous action sets
    - Tragedy of the commons
  - Duopoly models: Cournot and Bertrand
  - Comparison of duopoly models with Monopoly
- Today (January 18, 2007)
  - Duopoly models
    - Stackelberg - Comparison with Cournot, Bertrand, and Monopoly
  - Multistage games with observed actions
  - Subgame perfect equilibrium

Stackelberg Model

- Two competing firms, selling a homogeneous good
- The *marginal cost* of producing each unit of the good: c_1 and c_2
- Firm 1 moves first and decides on the quantity to sell: q_1
- Firm 2 moves next and after seeing q_1, decides on the quantity to sell: q_2
- $Q = q_1 + q_2$ total market demand
- The market price, P is determined by (inverse) market demand:
  - $P = a - bQ$ if $a > bQ$, $P = 0$ otherwise.
- Both firms seek to maximize profits

Stackelberg Model

- Q_j: the space of feasible q_j's, $j=1,2$
- Strategies of firm 2:

  $s^2: Q_1 \to Q_2$
- Strategies of firm 1: $q_1 \in Q_1$
- Outcomes and payoffs in pure strategies

  $(q_1, q_2) = (q_1, s^2(q_1))$

  $\pi^j(q_1, q_2) = [a-b(q_1+ q_2)- c_j]\ q_j$

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Stackelberg Model: Strategy of Firm 2

- Suppose firm 1 produces q_1
- Firm 2's profits, if it produces q_2 are:

  $\pi_2 = (P-c)q_2 = [a-b(q_1+ q_2)]q_2 \quad - c_2q_2$

  $= \text{(Residual) revenue} - \text{Cost}$
- First order conditions:

  $d\, p_2/dq_2 = a - 2bq_2 - bq_1 - c_2 =$

  $= \text{RMR} - \text{MC} = 0 \to$

  $q_2=(a-c_2)/2b - q_1/2 = R^2(q_1)$

  $s^2 = R^2(q_1)$ Strategy of firm 2

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Stackelberg Model: Firm 1's decision

- Firm 1's profits, if it produces q_1 are:

  $\pi_1 = (P-c)q_1 = [a-b(q_1 + q_2)]q_1 - c_1q_1$
- We know that from the best response of Firm 2:

  $q_2 = (a-c_2)/2b - q_1/2$
- Substitute q_2 into π_1:

  $\pi_1 = [a-b(q_1 + (a-c_2)/2b - q_1/2)]q_1 - c_1q_1$

  $= [(a+ c_2)/2-(b/2)\ q_1-c_1]q_1$
- From FOC:

  $dp_1/dq_1 = (a+ c_2)/2-b\ q_1 - c_1 = 0 \rightarrow$

  $q_1 = (a-2c_1+c_2)/2b$

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Stackelberg Equilibrium

- We have Firm 1's profits, if it produces q_1:

  $q_1 = (a-2c_1+c_2)/2b$

And firm 2's best response

  $q_2 = (a-c_2)/2b - q_1/2$

- Therefore:

  $q_2 = (a+2c_1-3c_2)/4b$
- If $c_1 = c_2 = c$

  $q_1 = (a-c)/2b$

  $q_2 = (a-c)/4b$

  $Q = 3(a-c)/4b$

Recall: $q^c_i = (a-c)/3b$

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Cournot vs. Stackelberg vs. Bertrand

	Bertrand	Stackelberg	Cournot	Monopoly
Price	c	(a+3c)/4	(a+2c)/3	(a+c)/2
Quantity	(a-c)/b	3(a-c)/4b ((a-c)/2b+ (a-c)/4b)	2(a-c)/3b	(a-c)/2b
Total Firm Profits	0	$3(a-c)^2/16b$	$2(a-c)^2/9b$	$(a-c)^2/4b$

Example: Stackelberg Competition

- $P = 130-(q_1+q_2)$, so a=130, b=1
- $c_1 = c_2 = c = 10$
- Firm 2: $q_2=(a-c_2)/2b - q_1/2 = 60 - q_1/2$
- Firm 1:
  - $\Pi_1 = [a-b(q_1+q_2)]q_1 - c_1q_1$
  - $\Pi_1 = [(a+c_2)/2-(b/2)q_1]q_1 - c_1q_1$
  - $\Pi_1 = [70-q_1/2]q_1 - c_1q_1$

  → q_1 = 60
- Market price and demand

  Q=90 P=40

Monopoly vs. Cournot vs. Bertrand vs. Stackelberg

	Bertrand	Stackelberg	Cournot	Monopoly
Price	10	40	50	70
Quantity	120	90 (60+30)	80	60
Total Firm Profits	0	2700 (1800+900)	3200	3600

- Firm profits and prices:
  Bertrand ≤ Stackelberg ≤ Cournot ≤ Monopoly

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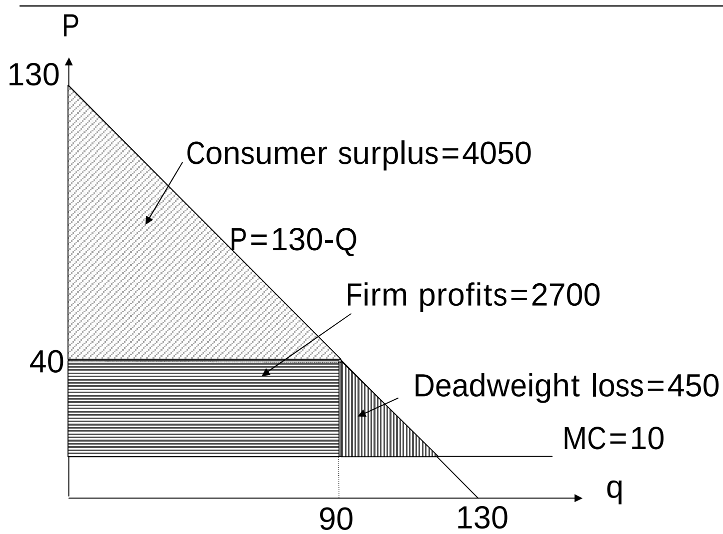
Stackelberg competition



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Monopoly vs. Cournot vs. Bertrand vs. Stackelberg

	Bertrand	Stackelberg	Cournot	Monopoly
Consumer surplus	7200	4050	3200	1800
Deadweight loss	0	450	800	1800
Total Firm Profits	0	2700 (1800+900)	3200	3600

Stackelberg and Information

- Does player 2 do better or worse in this Stackelberg game compared to the Cournot game?
- Does player 2 have more or less information in the Stackelberg game compared to the Cournot game?

Multi-Stage Games with Observed Actions

These games have "stages" such that

- Stage k+1 is played sequentially after stage k
- In each stage *k*, every player knows all the actions (including those by Nature) that were taken at any previous stage
  - Players can move simultaneously in each stage *k*
    - Some players may be limited to action set "do nothing" in some stages
    - Each player moves at most once within a given stage
- Players' payoffs are common knowledge

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Stackelberg game

- Stage 1
  - Firm 1 chooses its quantity q_1; Firm 2 does nothing
- Stage 2
  - Firm 2, knowing q_1, chooses its own quantity q_2; Firm 1 does nothing

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Multi-Stage Games with Observed Actions

h^k : History at the start of stage k

$h^k = (a^0, a^1, \ldots, a^{k-1}),\ k = 1, \ldots, K$

$A^i(h^k)$: Set of actions available to player i in stage k given history h^k

s^i : Pure strategy for player i that specifies an action $a \in A^i(h^k)$ for each k and each history h^k

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Finite games of perfect information

- A multistage game has ***perfect information*** if
  - each player knows all previous moves when making a decision
  - for every stage k and history h^k, exactly one player has a nontrivial action set, and all other players have one-element action set "do nothing"
- In a ***finite game of perfect information***, the number of stages and the number of actions at any stage are finite.
- Theorem (Zermelo 1913; Kuhn 1953): A finite game of perfect information has a pure-strategy Nash equilibrium

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Backward induction

Determine the optimal action(s) in the final stage K for each history h^K

For each stage j=K-1,...,1
- Determine the optimal action(s) in stage j for each possible h^j given the optimal actions determined for stages j+1,...,K.

The strategy profile constructed by backward induction is a Nash Equilibrium.

Each player's actions are optimal at every possible history.

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Example: Stackelberg competition

- $P = 130-(q_1+q_2)$, $c_1 = c_2 = c = 10$

Backward induction
- Firm 2 strategy: $s^2(q_1)= q_2= 60 - q_1/2$
- Firm 1 strategy: $q_1= 60$
- The outcome (60,30) is a Nash equilibrium (Stackelberg outcome)

Is (60,30) the unique equilibrium in this game?

Let's consider the Cournot equilibrium (40,40) for the Stackelberg game $s^2(q_1)= 40$ $q_1= 40$

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Stackelberg: other equilibria

- Suppose that player 2 commits to choosing the Cournot quantity no matter what
  - s_2(q_1)=q_2^C, for all q_1
- Then player 1's strategy should also be the Cournot quantity
  - q_1=q_1^C

- Cournot equilibrium (40,40) is also an equilibrium for the Stackelberg game! s2(q1)= 40 q1= 40

- Will this happen?
- Implications?

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Classroom exercise: Strategic investment

- Duopoly: Firm 1 and Firm 2
- Each firm has unit cost 2
- By paying \$f, Firm 1 can install new technology and reduce its unit cost to zero
- Once Firm 1's investment decision is observed, both firms simultaneously choose output levels q_1 and q_2 as in Cournot competition
- P=14-Q

Recall: Cournot best response

$q_1=(a-c_1)/2b - q_2/2$

$q_2=(a-c_2)/2b - q_1/2$

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Subgame-perfect equilibrium

- Denote:
  - G(h^k): game from stage k on with history h^k
  - For each player j, $s^j|h^k$ is the restriction of strategies s^j to the histories consistent with h^k
- A strategy profile s of a multistage game with observed actions is a *subgame-perfect equilibrium* if, for every h^k, the restriction $s|h^k$ is a Nash equilibrium of subgame G(h^k).

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Classroom exercise: Strategic investment

- Firm 1 does not invest
  - $q_1=(a-c_1)/2b – q_2/2 = 6 - q_2/2$
  - $q_2=(a-c_2)/2b – q_1/2 = 6 - q_1/2 \rightarrow (q_1, q_2)=(4,4)$

    Payoffs: (16,16)
- Firm 1 does invest
  - $q_1=(a-c_1)/2b – q_2/2 = 7 - q_2/2$
  - $q_2=(a-c_2)/2b – q_1/2 = 6 - q_1/2 \rightarrow$

    $(q_1,q_2)=(16/3,10/3)$

    Payoffs: (256/9-f,100/9)
- Firm 1 choice:

  Invest if 256/9-f>16, i.e., if f<112/9

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Example: Stackelberg competition

- $P = 130-(q_1+q_2)$, $c_1 = c_2 = c = 10$

By backward induction the outcome (60,30) is a subgame-perfect equilibrium

The outcome (40,40) is NOT subgame perfect, because the strategy $s^2(q_1) = 40$ does not induce a Nash equilibrium in stage 2 for player 2, for histories other than $q_1 = 40$

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Extensive form of a game

- The set of players
- The order of moves
- The players' payoffs as a function of the moves that were made
- The set of actions available to the players when they move
- Each player's information when he makes his move
- The probability distributions over any exogenous events (Nature)

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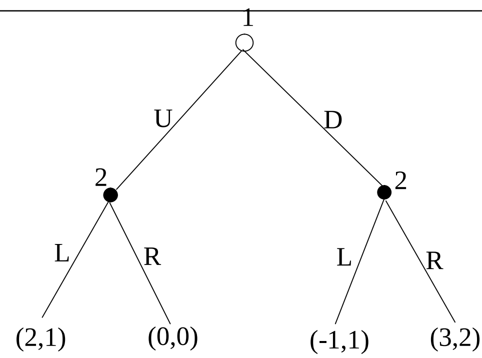
Example 1



Player 1 moves first. After observing player 1's action, player 2 moves
Player 1 action set: {U,D} Player 2 action set: {L,R}
Player 1 strategies: {U,D}
Player 2 strategies: {(L,L), (L,R), (R,L), (R,R)}
(Strategies specify a complete plan of action for all contingencies)

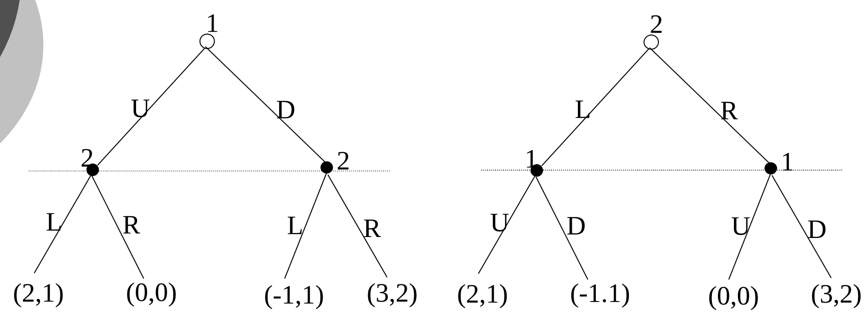
Normal form representation of extensive-form games

Player 2

Player 1	(L,L)	(L,R)	(R,L)	(R,R)
U	2,1	2,1	0,0	0,0
D	-1,1	3,2	-1,1	3,2

- Player 2's strategies correspond to a contingent plan made in advance
- We usually do not use normal-form representation of an extensive-form game

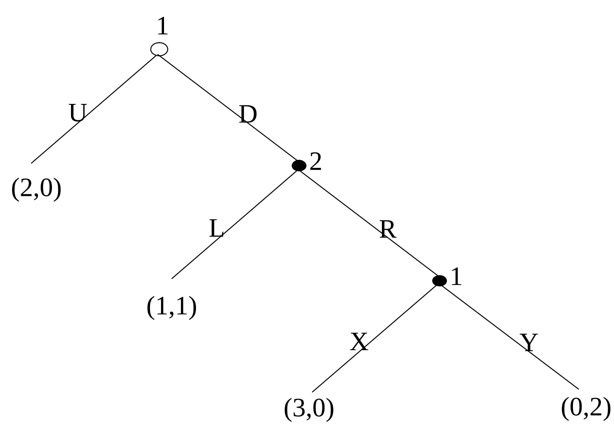
Example 2



Player 1 moves first, player 2 moves next. Player 2 does <u>not</u> know player 1's action when he chooses his action

Player 2 moves first, player 1 moves next. Player 1 does <u>not</u> know player 2's action when he chooses his action

Classroom exercise



Example to Draw

- Player 1 chooses an action from the feasible set {L,R}
- Player 2 observes player 1's action and then chooses an action from the feasible set {L',R'}
- Player 3 observes whether or not the history of actions is (R,R') and then chooses an action from the feasible set {L'',R''}

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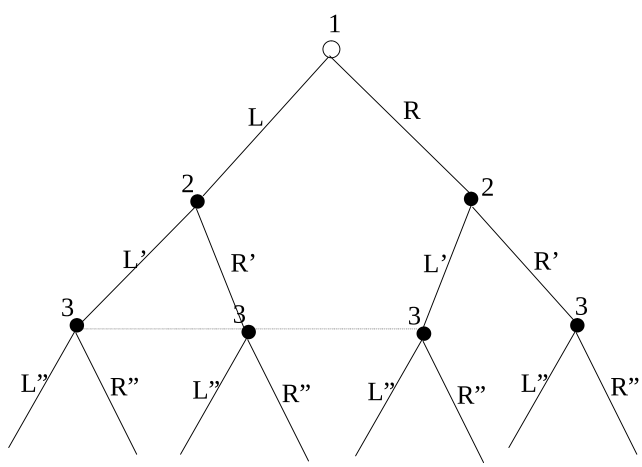
Example (cont.)



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