Recap

- Last class (January 25, 2007)
  - Two-stage prisoner's dilemma
  - Finitely repeated games
    - If there is a unique NE in the static game, it is the NE of the repeated game
    - If there are multiple NE in the static game, the repeated game may have NE that are not NE in the static game
  - Discount payoffs
- Today (January 30, 2007)
  - Infinitely repeated games
    - Prisoner's dilemma
    - Friedman's Theorem
    - Repeated Cournot game

Proof of Friedman's Theorem

- Let $(a_{e1}, a_{e2}, \ldots, a_{en})$ be the Nash equilibrium of G that yields the equilibrium payoffs $(e^1, e^2, \ldots, e^n)$.
- Let $(a_{x1}, a_{x2}, \ldots, a_{xn})$ be the collection of actions that yields the equilibrium payoffs $(x^1, x^2, \ldots, x^n)$.
- Trigger strategy for player i:
  - Play a_{xi} in the first stage. In the t^{th} stage, if the outcome of all t-1 preceding stages has been $(a_{x1}, a_{x2}, \ldots, a_{xn})$ then play a_{xi}; otherwise, play a_{ei}.
- Show that the trigger strategies induce a NE if δ is sufficiently large
- Show that the equilibrium is subgame perfect

Proof of Friedman's Theorem (cont.)

- Suppose all players other than player i use the trigger strategy.
- Best response of player i in stage t:
  - If the outcome of the previous stage differs from (a_{x1}, a_{x2}, ... , a_{xn})
    - Play a_{ei} forever
  - If the outcomes of all previous stages are (a_{x1}, a_{x2}, ... , a_{xn})

$$\max_{a_i \in A_i} p^i(a_{x1},\dots,a_{x,i-1},a_i,a_{x,i+1},\dots,a_{xn}) = d^i$$

$$d^i \geq p^i(a_{x1},\dots,a_{x,i-1},a_{xi},a_{x,i+1},\dots,a_{xn}) > p^i(a_{e1},\dots,a_{en}) = e^i$$

Proof of Friedman's Theorem (cont.)

- If player i deviates in stage t by choosing a_{di}:
  - Payoff in stage t: d^i
  - Payoff in future stages:
    - $\delta e^i + \delta^2 e^i + \dots = \delta e^i / (1-\delta)$
  - Total (discounted) payoff: $V^i = d^i + \delta e^i / (1-\delta)$
- If player i plays a_{xi} in stage t:
  - Receive a payoff x^i in this stage, face the same game in the next stage.

  $V^i = x^i + \delta V^i \rightarrow V^i = x^i / (1-\delta)$.
- Playing x^i is optimal if and only if

  $x^i / (1-\delta) \geq d^i + \delta e^i / (1-\delta) \rightarrow \delta \geq (d^i - x^i)/(d^i - e^i)$

Proof of Friedman's Theorem (cont.)

- It is Nash equilibrium for all players to play the trigger strategy if and only if
  $\delta \geq \max_i \ (d^i - x^i)/(d^i - e^i)$
- Subgame perfectness:
  - If the outcome of the previous stage differs from $(a_{x1}, a_{x2}, \ldots, a_{xn})$
    - Play a_{ei} forever
  - If the outcomes of all previous stages are $(a_{x1}, a_{x2}, \ldots, a_{xn})$
    - Play the trigger strategy

Trigger Strategies

- What characterizes this kind of trigger strategy?
- What other strategies might be possible?
- Advantages? Disadvantages?

Repeated Cournot Game

- Cournot stage game
  - Two competing firms, selling a homogeneous good
  - The *marginal cost* of producing each unit of the good: c
  - The market price, P is determined by (inverse) market demand:
    - $P=a-Q$ if $a>Q$, $P=0$ otherwise.
  - Each firm decides on the quantity to sell (market share): q_1 and q_2
  - $Q= q_1+q_2$ total market demand
  - Both firms seek to maximize profits
- Unique NE of the stage game: $q^C=(a-c)/3$ $Q= 2(a-c)/3$
- Monopoly quantity: $q^M=(a-c)/2$

Repeated Cournot Game (cont.)

- The stage game is repeated infinitely many times
- The firms have discount factor δ
- Trigger strategy
  - Produce half the monopoly quantity, $q^M/2$, in the first stage. In the t^{th} stage, produce $q^M/2$ if both firms have produced $q^M/2$ in all previous stages; otherwise, produce q^C.
- Show that the trigger strategy induces a subgame perfect NE.

Repeated Cournot Game (cont.)

- Profit of one firm
  - If both produce $q^M/2$: $(a-c)^2/8 = \pi^M/2$
  - If both produce q^C: $(a-c)^2/9 = \pi^C$
- Best response of firm i:
  - If the last stage outcome is other than $(q^M/2, q^M/2)$
    - Play q^C forever
  - If all previous stages' outcomes are $(q^M/2, q^M/2)$
    - Deviate

      $\max\ (a-q_i-q^M/2-c)\ q_i \rightarrow$

      $q_i = 3(a-c)/8 \qquad \pi^D = 9(a-c)^2/64$

      $V^i = \pi^D + \pi^C \delta /(1-\delta)$
    - Play $q^M/2$

      $V^i = \pi^M/2 + \delta V^i \rightarrow V^i = \pi^M /2(1-\delta)$

Repeated Cournot Game (cont.)

- Profit of one firm
  - If both produce $q^M/2$: $(a-c)^2/8 = \pi^M/2$
  - If both produce q^C: $(a-c)^2/9 = \pi^C$
- Best response of firm i:
  - If the last stage outcome is other than $(q^M/2, q^M/2)$
    - Play q^C forever

  - If all previous stages' outcomes are $(q^M/2, q^M/2)$
    - Deviate: $V^i = \pi^D + \pi^C \delta /(1-\delta)$
    - Play $q^M/2$: $V^i = \pi^M /2(1-\delta)$
    - Playing the trigger strategy is NE iff

      $\pi^M /2(1-\delta) \geq \pi^D + \pi^C \delta /(1-\delta) \rightarrow \delta \geq 9/17$

Repeated Cournot game

- Trigger strategy
  - Produce half the monopoly quantity, $q^M/2$, in the first stage. In the t^{th} stage, produce $q^M/2$ if both firms have produced $q^M/2$ in all previous stages; otherwise, produce q^C.
- Playing the trigger strategy is SPNE iff $\delta \geq 9/17$

What if $\delta < 9/17$?

Repeated Cournot Game (cont.)

- Given a fixed δ:
- Trigger strategy
  - Produce q^*, in the first stage. In the t^{th} stage, produce q^* if both firms have produced q^* in all previous stages; otherwise, produce q^C.
- Profit of one firm
  - If both produce q^*: $(a-2q^*-c)\, q^* = \pi^*$
  - If both produce q^C: $(a-c)^2/9 = \pi^C$
  - If firm j produces q^* and firm i deviates:

    $\max\ (a- q_i- q^*-c)\, q_i \rightarrow q_i = (a- q^*-c)/2$

    $\pi^D = (a- q^*-c)^2/4$

Repeated Cournot Game (cont.)

- Best response of firm i:
  - If the last stage outcome is other than (q^*, q^*)
    - Play q^C forever
  - If all previous stages' outcomes are (q^*, q^*)
    - Deviate: $V^i = \pi^D + \pi^C \delta / (1-\delta)$
    - Play q^*: $V^i = \pi^* + \delta V^i \rightarrow V^i = \pi^* / (1-\delta)$
    - Playing the trigger strategy is NE iff

    $\pi^*/(1-\delta) \geq \pi^D + \pi^C \delta / (1-\delta)$

    Substitute and solve for q^*:

    $q^* = (9-5\delta)(a-c)/3(9-\delta)$

Recall: $q^C=(a-c)/3$ $q^M=(a-c)/2$

Repeated Cournot Game (cont.)

- Playing the trigger strategy is NE iff $\pi^*/(1-\delta) \geq \pi^D + \pi^C \delta / (1-\delta)$
- Substitute and solve for min q^*: $q^* = (9-5\delta)(a-c)/3(9-\delta)$
- As $\delta \rightarrow 0$, $q^* \rightarrow q_C$
- As $\delta \rightarrow 9/17$, $q^* \rightarrow q_m/2$