First-Price Sealed-Bid Auction

- Two bidders, one good
- Bidder i’s valuation for the good is \( v_i \), is known only by bidder i. Valuations are independently and uniformly distributed on \([0,1]\).
- Each bidder i submits a nonnegative bid \( b_i \). The higher bidder wins and pays his bid. Other bidder pays and receives nothing.
- In case of a tie, the winner is determined by a coin flip
- Bidder i’s payoff, if wins and pays \( p \), is \( v_i - p \)
- Bidders are risk-neutral
- All of this information is common knowledge

First-Price Sealed-Bid Auction

- Action spaces
  - \( A_1 = A_2 = [0,\infty) \)
- Type spaces
  - \( T_1 = T_2 = [0,1] \)
- Beliefs
  - \( p_1(t_2|t_1) = p_1(t_2) \)
  - \( p_2(t_1| t_2) = p_2(t_1) \)
- Player i’s (expected) payoff function

\[
\pi_i(b_1, b_2; v_1, v_2) = \begin{cases} 
  v_i - b_i, & \text{if } b_i > b_j \\
  (v_i - b_i)/2, & \text{if } b_i = b_j \\
  0, & \text{if } b_i < b_j 
\end{cases}
\]
First-Price Sealed-Bid Auction

- Strategy for player $i$: $b_i(v_i)$
- Strategies $(b_1(v_1), b_2(v_2))$ are a Bayesian Nash equilibrium if for each $v_i$ in [0,1], $b_i(v_i)$ solves
  \[ \max (v_i - b_i) \Pr\{b_i > b_j(v_j)\} + (v_i - b_i) \Pr\{b_i = b_j(v_j)\}/2 \]

- What type of strategy might make sense?

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First-Price Sealed Bid Auction

- Let’s see if there is a linear equilibrium $b_i(v_i) = a_i + c_i v_i$, $i = 1, 2$

- Assuming player $j$ adopts the strategy $b_j(v_j) = a_j + c_j v_j$, player $i$’s best response:
  \[ \max (v_i - b_i) \Pr\{b_i > b_j(v_j)\} = (v_i - b_i) \Pr\{b_i > a_j + c_j v_j\} \]

- Player $i$ knows:

<table>
<thead>
<tr>
<th>$a_j$</th>
<th>$a_j + c_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pdf of $b_j$</td>
<td></td>
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Example strategies

Recall for values in range that Uniform variables have pdf $= 1/(b-a)$ and cdf $= (x-a)/(b-a)$.  

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First-Price Sealed-Bid Auction

- Given \( b_i(v_j) = a_i + c_j v_j \), player i should not bid below player j’s minimum or above player j’s maximum so
  \[ b_i, a_j, \quad b_i \cdot a_j + c_j \]

- We expect a higher-value type to bid more than lower ones
  \[ c_j, 0 \]

- We know \( \text{Prob}(b_i > a_i + c_j v_j) = \text{Prob}(v_j < (b_i - a_i)/c_j) \)
  \[ = (b_i - a_i)/c_j \] From cdf for
  \[ v_j = (x - 0)/(1 - 0) \]

- Player i’s objective:
  - \( \max (v_i - b_i) \text{Prob}(b_i > a_i + c_j v_j) = (v_i b_i - b_i^2 + b_i a_i - v_i a_j)/c_j \)
  - From FOC, \( b_i = (v_i + a_i)/2 \) (and SOC okay)

- Player i’s best response (thus far):
  - \( b_i = a_i \) if \( v_i \leq a_i \) (from constraint \( b_i, a_i \)),
  - \( b_i = (v_i + a_i)/2 \) otherwise

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First-Price Sealed-Bid Auction

- Player i’s best response
  \( b_i = a_i \) if \( v_i \leq a_i \), \( b_i = (v_i + a_i)/2 \) otherwise

- Can \( a_i \) be
  - Between 0 and 1?
    - For some values, \( v_i \cdot a_i \), so not linear
  - Greater than or equal to 1?
  - Since \( c_j, 0 \), \( b_i(v_j) = a_j + c_j v_j \geq 1 \)
    - Then \( b_i(v_j), v_i! \)
  - Less than or equal to zero?
    - \( b_i(v_i) = (v_i + a_i)/2 \)

We have \( a_j \leq 0, b_i = a_i + c_i v_i \) (from linear form)
\[ = a_i/2 + (v_i) (\text{best response to } j) \]
\[ \rightarrow a_i = a_j/2 \quad c_i = 1/2 \]
First-Price Sealed-Bid Auction

- Player i’s best response
  \[ a_i \leq 0, \quad a_i + c_i v_i = a_j / 2 + 1/2(v_i) \rightarrow a_i = a_j / 2, \quad c_i = 1/2 \]
- Player j’s best response
  \[ a_j \leq 0, \quad a_j + c_j v_j = a_i / 2 + 1/2(v_j) \rightarrow a_j = a_i / 2, \quad c_j = 1/2 \]

We have
\[ a_i = a_j = 0, \quad c_i = c_j = 1/2 \]
and
\[ b_i(v_i) = v_i / 2 \quad i = 1, 2 \]
Each player bids half his/her valuation in a linear equilibrium. (If players’ strategies are strictly increasing and differentiable, this is the unique symmetric equilibrium)

A Double Auction Example

- A seller and a buyer have private valuations \( v_s \) and \( v_b \)
  - Assume drawn from independent uniform distributions on [0,1]
- Seller names asking price \( p_s \); buyer simultaneously names offer price \( p_b \)
- If \( p_b \geq p_s \), then trade occurs at price \( p = (p_b + p_s) / 2 \); if \( p_b < p_s \) then no trade occurs
- Utilities if trade occurs are \( p-v_s \) and \( v_b-p \) (and 0 otherwise)
- Find strategies that specify the price to offer (or demand) for each of the other party’s valuations
Double Auction Example

- A pair \( \{p_b(v_b), p_s(v_s)\} \) is a BNE if below true:

- For buyer, for each \( v_b \) in \([0,1]\), \( p_b(v_b) \) solves
  - 1. \( \max_{p_b} [v_b - (p_b + \varepsilon)/2] \times \text{Prob}(p_b, p_s(v_s)) \)
    - where \( \varepsilon = E[p_s(v_s) | p_b, p_s(v_s)] \) is the expected price the seller will demand, conditional on demand being less than the buyer’s offer of \( p_b \)

- For seller, for each \( v_s \) in \([0,1]\), \( p_s(v_s) \) solves
  - 2. \( \max_{p_s} [(p_s + \varepsilon)/2 - v_s] \times \text{Prob}(p_b(v_b), p_s) \)
    - where \( \varepsilon = E[p_b(v_b) | p_b(v_b), p_s] \)

Double Auction

- Trades will never occur when a seller’s valuation is higher than the buyer’s valuation.

- What are several simple strategies?
Double Auction

Consider a 1-price equilibrium

- For any value $x$ in $[0,1]$,
  - buyer offers $x$ if $v_b$ · $x$ and 0 otherwise
  - seller demands $x$ if $v_s$ · $x$ and 1 otherwise

Trade here would be "efficient" but does not occur

Is there a linear equilibrium?

- If seller’s strategy is $p_s(v_s) = a_s + c_s v_s$
  - $v_s$ is U[0,1] and $p_s$ is U[$a_s$, $a_s + c_s$]

To determine buyer’s response we need

1. $\text{Prob}\{p_b, p_s(v_s)\}$ which is $\text{Prob}\{(p_b - a_s)/c_s, v_s\}$
   - $\text{CDF} \Rightarrow \text{Prob} = (p_b - a_s)/c_s.$
2. $\text{EPS} = \mathbb{E}[p_s(v_s)|p_b, p_s(v_s)] = (a_s + p_b)/2.$
Double Auction

- Then from buyer’s function (1):
  - \( \max_{p_b} \left[ v_b - \frac{p_b + (a_s + p_b)/2}{2} \right] \frac{(p_b - a_s)}{c_s} \)
  - FOC \( p_b = \frac{2v_b}{3} + \frac{a_s}{3} \) gives buyer’s response (which is linear)
  - And SOC okay

Double Auction

- If buyer’s strategy is \( p_b(v_b) = a_b + c_b v_b \),
  - \( v_b \) is \([0,1]\) and \( p_b \) is \([a_b, a_b + c_b]\)
- To find seller’s best response we need
  1. \( \text{Prob}\{p_b(v_b), p_s\} \) which is \( \text{Prob}\{a_b + c_b v_b, p_s\} \)
     = \( \text{Prob}\{v_b, (p_s - a_b)/c_b\} \) \( \text{CDF} \Rightarrow \text{Prob} = 1 - \frac{(p_s - a_b)/c_b}{(c_s + p_s + a_b)/c_b} \)
  2. \( \text{EPB} = E[p_b(v_b)|p_b(v_b), p_s] \)
     = \( (p_s + a_b + c_b)/2. \)
Double Auction

- Then from seller’s function (2):
  - \( \max_{p_s} \left[ \{p_s+(p_s+a_b+c_b)/2\}/2-v_b \right] + (a_b+c_b-p_s)/c_b \)
  - FOC \( p_s = 2v_s/3 + 1/3(a_b+c_b) \) is seller’s best response (which is linear)
  - And SOC okay

- Now the best response functions are
  - \( p_b = 2v_b/3 + a_b/3 \)
  - \( p_s = 2v_s/3 + 1/3(a_b+c_b) \)
  - So \( c_b = 2/3 \) and we can solve for \( a_b \).

Double Auction

- 2 BR functions together give us:
  - \( p_b(v_b) = 2/3 v_b + 1/12 \)
  - \( p_s(v_s) = 2/3 v_s + 1/4 \)
- Trade occurs only if \( p_b \geq p_s \) or if \( v_b \geq v_s + 1/4 \)
Double Auction

- Compare trades in 1-price and linear equilibrium
- Linear may “dominate” 1-price BNE
  - (for uniform valuations, gives higher expected gains than any other BNE)

Revelation Principle

- There may be many different “mechanisms” to achieve a goal
  - Direct mechanism is a game where a player’s only action is to submit a (possibly dishonest) claim about his or her type
  - A direct mechanism in which truth-telling is a Bayesian Nash equilibrium is called “incentive-compatible”
    - Players should be “willing to play”

- Myerson (1979):
  - Any Bayesian Nash equilibrium of any Bayesian game can be represented by an incentive-compatible direct mechanism