

Optimal Timing of Switches between Product Sales for Sports and Entertainment Tickets

Matthew J. Drake
Schools of Business
Duquesne University
Pittsburgh, PA 15282-0180
drake987@duq.edu

Serhan Duran • Paul M. Griffin • Julie L. Swann
H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0205
sduran@isye.gatech.edu • pgriffin@isye.gatech.edu • jswann@isye.gatech.edu

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Like airlines and hotels, sports teams and entertainment venues can benefit from revenue management efforts for their ticket sales. Teams and entertainment venues usually offer bundles of tickets early in their selling horizon and put single-event tickets on sale at a later date. All of these organizations must determine the best time to offer individual tickets because both types of ticket sales consume the same fixed inventory. We model the seller's optimal *a priori* timing decision for switching product offerings from selling bundles to individual tickets in order to maximize revenue when bundle and single-ticket customers each arrive according to independent, non-homogeneous Markovian death processes with a linear death rate that can vary over time. We characterize the circumstances in which it is optimal for the seller to practice mixed bundling and when the seller should only sell bundles or individual tickets. We establish comparative statics for the optimal timing decision for the special case of constant customer arrival rates as model parameters such as bundle size, marginal revenues, and customer arrival rates vary. We extend our results to find the optimal time for offering products with different demand characteristics, and we apply the model to a data set obtained from the sports industry.

1 Introduction

Revenue management has its origins in the transportation industry, and extensive research in the field has been accomplished in the past two decades. Much of this research has addressed optimal and heuristic pricing or inventory policies for selling a fixed number of airline tickets for a specific flight in order to maximize the expected revenue. Extensions have been developed for products with similar demand characteristics such as hotel reservations and car rentals. A decidedly smaller library of research exists concerning revenue management for non-travel industries. In particular, the market for sports and entertainment event tickets has not received a great deal of attention.

There are some important differences between the travel and entertainment industries with respect to revenue management decisions that necessitate separate modeling frameworks. Passengers typically have a reason to travel to a specific location; whereas, demand for sporting and entertainment events is not derived from the demand for other products or services. Airlines may also alter their prices dynamically in response to realized demand, while teams and entertainment venues typically publish a price schedule before the beginning of the season and stick to it for the entire year. These organizations may offer discounts during the season for specific events or customer groups. Many consumers are willing to purchase a bundle of tickets to different sports or entertainment events that occur over the course of several months; the sale of these bundles and individual tickets, which are the products sold by sports and entertainment organizations, is the focus of our study.

Many spectators at sporting events are willing to consider purchasing tickets to multiple games at one time. This enables the sports teams to offer their tickets in mixed bundles—selling goods both individually as well as in packages. Teams often put season tickets on sale first, using the demand for popular events to drive the sale of these bundles. Georgia Tech, for example, is able to induce fans to purchase partial- or full-season packages in order to secure coveted tickets to the bi-annual home football game with the rival University of Georgia or other popular opponents such as the University of Notre Dame.

In this paper we analyze an optimal timing problem that is one aspect of the complex revenue management environment faced by sports franchises and entertainment organizations. We model the seller's decision of the time within the selling horizon at which she should make single tickets available or, equivalently, when she should promote the individual tickets. We consider a static decision environment because many organizations publish their schedule of ticket availability early in the selling horizon; e.g., many Major and Minor League Baseball teams announced their single-ticket availability for the 2006 season before the tickets went on sale (MLB, 2006). The Jacksonville Jaguars (2006) of the National Football League also publicized the sale date of individual tickets several weeks in advance. Cultural organizations such as New York's Metropolitan Opera (2005), Atlanta's Fox Theatre (2006), and the Georgia Institute of Technology's Robert Ferst Center for the Arts (2005) make the timing decision of offering single-event tickets prior to the selling season as well and announce the dates to the public far in advance.

Although a static timing decision may be restrictive, the fact that many different organizations publish their ticket sales dates in advance indicates that they realize some benefit by doing so. Benefits include the reduction of expedited printing costs for advertising materials, more time for

workforce planning, and a smaller amount of worker uncertainty by settling the sale dates sooner. An earlier commitment to the sales date allows the organization to advertise the dates for a longer period of time. The proliferation of this practice may also be driven by customer behavior and expectations. Loyal customers who buy packages may also react negatively to an unexpected reduction in the sales period, and the organizations do not want to risk an adverse reaction because the package buyers are their most important customers. This insight coincides with the feedback that we have obtained from talking with several sports and entertainment organizations.

In our model we assume that customers arrive according to pure, non-homogeneous Markovian death processes with linear death rates that can vary over time. The demand processes for bundles and individual tickets are assumed to be independent from each other, and the organization makes the timing decision in order to maximize the revenue obtained before the performance season begins. For many possible parameter values, the seller should not practice mixed bundling; instead, she should offer either only season ticket or individual tickets for the entire selling season. The fact that the customer arrival rate is allowed to change over time precludes us from establishing a closed-form solution for the optimal timing decision, but we do attain a closed-form solution in the special case where the arrival rate is constant over the entire selling horizon. We determine how this optimal policy changes in the special case as a function of other model parameters such as the bundle size, marginal revenues, and customer arrival rates. We initially assume that demand is the same for all events, but we extend our results for the general model to the scenario where event tickets have different demand characteristics.

While we develop this model in the context of a sports or entertainment ticket market, the framework can be extended to other industries where blocks of goods are sold in different quantities at different points in time. One possibility is the sale of hotel rooms first to groups and then to individual guests. Another application would be the allocation of fixed capacity by a contract manufacturer or third-party warehouse.

The remainder of the paper is organized as follows. The next section contains the literature review. Section 3 presents the general model for a single timing decision in which the seller offers either a bundle or individual units from a fixed quantity of homogeneous inventory. Section 4 contains an extension of the general model to the bundling of event tickets having different demand characteristics. Section 5 details empirical analysis of a data set comprised of ticket sales transactions for the 2003 Georgia Tech home football season, and the conclusions of the study are provided in Section 6 along with several future research directions.

2 Literature Review

The problem we study relates to the literature about commodity bundling, sports operations, and timing problems in pricing, in addition to existing research about ticket pricing. The economics and marketing streams of literature contain several studies about ticket pricing for entertainment events. Leslie (2004) used data from a Broadway production to estimate a discrete-choice random utility model of ticket sales with three different seat qualities under various pricing mechanisms. Rosen and Rosenfield (1997) develop an economic model with two seat qualities. They characterize revenue-maximizing prices given the quantities of each seat quality and then determine the quantity of seats to allocate to each quality based on this optimum pricing policy. Neither of these studies considers the bundling of tickets (e.g., season tickets), and they do not incorporate a temporal element in their pricing decisions.

Venkatesh and Mahajan (1993) determine the optimal pricing policies for general products and services under the three types of bundling: pure commodities (selling only individual goods), pure bundling (selling only bundles of goods), and mixed bundling (selling goods both separately and within packages). Other variations on pricing goods sold in bundles under various competitive conditions are presented in Venkatesh and Kamakura (2003), Bakos and Brynjolfsson (2000), Salinger (1995), McAfee et al. (1989), and Dansby and Conrad (1984). Again, there is no consideration of offering different bundles at different points in time; these studies are concerned solely with the pricing decisions of the bundles.

DeGraba and Mohammed (1999) develop a model of intertemporal mixed bundling with two types of consumers in a concert ticket environment with deterministic demand with the goal of setting prices. In their model bundles of tickets to two shows are sold first, and then single-performance tickets are sold at a later, exogenously-determined point in time. They show that the concert promoter can offer a price for the bundle of two shows that exceeds the price of the individual tickets if they can induce fear in the consumers that they will be sold out of seats by the time the single-event tickets go on sale. Their main concern is the pricing decision, while ours is the timing of the offering of the smaller bundle.

A number of recent articles and conference presentations have highlighted the application of operations research and management science techniques to the problems faced by professional sports teams. Much of this research, however, has been in developing improved game schedules that satisfy an extremely large set of preference constraints and (in some studies) minimize travel distance between matches through the use of large-scale integer programming models (see, e.g.,

Bean and Birge (1980) and Nemhauser and Trick (1998)) or in predicting players' and teams' performance through statistical analysis (see, e.g., Kvam and Sokol (2006)). An additional body of research in statistics and economics investigates influential factors in teams' yearly attendance (see, e.g., Boyd and Boyd (1998) and Cochran (2002)). Fewer papers in operations research have focused on improving the service operations of these organizations.

Optimal timing problems have been studied in several different research contexts; one significant problem concerns the optimal timing of pricing policies. (See Elmaghraby and Keskinocak (2003) and Chan et al. (2004) for detailed reviews of the dynamic pricing literature.) Feng and Gallego (1995) provide a stochastic model of the timing of a single price change, either a markup or a markdown, and show that a time threshold policy that is dependent on the remaining inventory is optimal. Petruzzi and Monahan (2003) characterize the optimal time for a firm to terminate the primary selling season (in terms of a sales target) in a discrete time setting where demand in a given period is an increasing function of the cumulative demand until that point in time. Our model differs from all of these in that it considers bundles of products where demand is dependent on the remaining inventory level at the time of the transaction.

As noted by Yuen (2002) and Farley (2003), there has been very little written about the purchase of bundles of products in the revenue management literature. This is not too surprising, however, since much of the research in revenue management has been conducted in the travel industry. Purchases of multiple tickets in the travel industry usually consist of a block of seats or hotel rooms for a given flight or night; individuals do not usually purchase multiple itineraries at once over the course of several months as they do in the sports and entertainment industry. Li (2002) states that his model can be used as a bound on the value of offering a block of seats to a tour operator, but no formal details are provided. One travel product that exhibits some of the characteristics of a season ticket is the "flexible product" studied by Gallego and Phillips (2004) in which the individual purchases options for several flights and then is assigned to specific flights by the carrier at a later point in time.

To summarize our contributions to the existing literature, we consider the timing decision for offering different products (i.e., bundles and single-event tickets) in contrast to the optimal timing studies that model price markdowns and markups for a single product. We are primarily concerned with the *time* at which to offer particular bundles unlike the commodity bundling literature, which analyzes the optimal pricing decision for different bundles. Another contribution of our research is the employment of a demand model in which the quantity demanded is allowed to change over time. The rationale for this characteristic of demand is explained in Section 3.1.

3 General Model

Many sports and entertainment venues advertise two types of ticket offerings: single-event tickets and season ticket packages.¹ Season tickets are generally sold first in the selling season, and individual-event tickets are made available all together on a single later date. In this section we develop a model that a venue can use to determine the optimal *a priori* time to start selling the single-event tickets.

3.1 Notation and Assumptions

For generality of application, we shall subsequently refer to all tickets as “units” and ticket packages as “bundles.” We consider a single seller who faces a selling season of T time units in length with a fixed k units of inventory. There is no possible inventory replenishment during the selling season because that would entail adding capacity to the stadium, arena, or venue. The seller can offer these goods as single units or bundles consisting of a units each, and we assume that k is an integer multiple of a to avoid complications from units that cannot be allocated to bundles. (This assumption is not very restrictive for large values of k , which is true at most large organizations.) As a result the firm can sell a maximum of k/a bundles or k individual units or any combination of the two. Bundles of units yield a unit revenue of r_B (and a total revenue of $r_B a$), and individual units sell for r_S each. The seller must determine the *a priori* optimal time $u \in [0, T]$ to begin selling individual units given that she began the horizon by selling bundles. The time at which individual tickets go on sale is normally the same for seats at all price levels. Because we are modeling only one switching time, r_B and r_S can be thought of as *average* prices over all price levels. An interesting extension of this model would allow tickets at different price levels or for different events to go on sale at different points in time, but this would necessitate a new model.

We initially assume that these k units have homogenous demand characteristics. Admittedly, many factors affect the demand for tickets to sporting events and entertainment performances, so it might seem simplistic to aggregate the tickets into a set of inventory with homogeneous demand characteristics. This assumption, though, enables us to isolate the effects of the independent demand processes and the prices on the optimal timing policy in the basic model. We relax this assumption in Section 4 and develop a model for inventory with different demand characteristics.

All the professional and college sports teams and cultural organizations with which we have had

¹These “season” tickets need not necessarily comprise the entire season of performances. Each theater or hall uniquely determines the number of events that constitutes a season ticket. The criterion we employ for a season ticket package is that it contains admission to more than one event.

discussions have emphasized the importance of selling ticket packages as opposed to single-game tickets. Many non-profit organizations require their season-ticket buyers to make a significant donation for the privilege of buying tickets early in the selling season. Teams in the National Football League also require patrons to purchase a Personal Seat License (PSL) that affords them the opportunity to purchase season tickets at all. Major college football teams such as Georgia Tech have also adopted this policy recently. Anil Malhotra, Partner at Analytix Consulting, an organization that focuses on revenue management practices in the arts, summed the importance of season-ticket holders as follows: “Season-ticket holders are the most valuable segment for several reasons. First, season-ticket holders generate the most ticket revenue per household as a result of their increased ticket commitment. Second, they have a greater likelihood of donating, purchasing merchandise, and spreading positive word of mouth that can motivate others to buy tickets. And finally, season-ticket buyers cost less per dollar of revenue to re-acquire the following season than do single ticket buyers” (Malhotra, 2006). Because season-ticket holders are more valuable to these organizations even though they might purchase tickets at a nominal unit discount, we extend the definition of “unit revenue” to “unit benefit.”

There are also markets in which goods *are* typically sold at no discount or together at a higher price than the sum of their individual prices. In 2003 the Georgia Tech Athletic Department charged the same unit price for season tickets to its football games as it does for individual sales; New York’s Metropolitan Opera follows a similar pricing policy.² DeGraba and Mohammed (1999) develop a model where concert venues can charge this higher bundle price if some consumers feel that there is a chance that their desired event will be sold out before single-event tickets go on sale. The venue can charge the concert-goers extra for the time utility of securing their seats well in advance of the actual event. In markets for collectibles such as antiques or sports cards, sellers routinely charge more (and often substantially more) for a complete set of items than they would for the goods individually. Consequently, we consider scenarios in which the unit revenue from tickets sold in packages is at least as large as that of individual ticket sales.

As is the case for many consumer products, sports and entertainment tickets embody the maxim that all are not created equal. Tickets to different events obviously have different values in the minds of the consumers based on the popularity of the participants, but even tickets to the same event at the same price level can be valued very differently. Most, if not all, venues sell the seats remaining that are closest to the field or stage whenever they make a transaction. Consequently, as more and

²In 2006 Georgia Tech began charging a price premium in the form of donations to the athletic department for season ticket holders in select seat locations.

Table 1: Notation (in order of discussion)

T	The end of the selling season
k	The number of units available to be sold before time T
a	The number of units in each bundle
r_B	Unit price of goods in the bundle (containing a units)
r_S	Unit price of goods sold individually
u	The time to begin selling individual units (the decision variable)
$X_B(t)$	The pure death process for bundles, which is dependent on time, t
$X_S(t)$	The pure death process for single units, which is dependent on time, t
$\mu_n^B(t)$	Death rate for bundles as a function of time, t , and remaining inventory level, n
$\mu_n^S(t)$	Death rate for single tickets as a function of time, t , and remaining inventory level, n

more seats are sold at a given price, the ones that remain are less valuable than the ones already sold because they are further away from the action. Many organizations such as the Metropolitan Opera (2006) advertise early ticketing priority to encourage early commitments or large donations. A demand rate that is directly related to the inventory level can serve as a proxy for the quality degradation of the remaining inventory as goods are sold. This relationship between the demand rate and the remaining inventory level is evident in the empirical data set analyzed in Section 5.

In some cases, however, demand may increase as the remaining inventory level falls. This could occur when the organization launches a major promotion during the selling season or when consumers fear that a popular event will sell out. The latter situation is exacerbated by the fact that many organizations offer online ticketing services where consumers can easily view the number of remaining tickets.³ To capture the relationship between demand, inventory level, and time, we model customer arrivals according to a special kind of non-homogeneous birth and death process, the pure Markovian death process with a linear death rate that is dependent on time. Because a sports or entertainment organization has a fixed amount of tickets to sell for a specific event season, the birth rates are zero. By using this function, we can model dependencies of demand on time or inventory; we can also find an explicit, closed-form solution in some cases.

The two Markovian death processes with linear death rates corresponding to demand for bundles and individual tickets are denoted by $\{X_B(t) : t \in [0, T]\}$ and $\{X_S(t) : t \in [0, T]\}$, respectively. $X_B(t)$ denotes the number of bundles remaining after selling bundled units from time period 0 to t . By definition, $X_B(0) = k/a$, and $X_B(t) = 0$ means that the seller has sold out of inventory by time t . The only values that $X_B(t)$ can assume are $k/a, k/a - 1, \dots, a, 0$. $X_S(t)$ denotes the number of

³Even organizations that do not have this online capability will generally provide this information to customers over the telephone.

units remaining after selling individual units for t time periods. $X_S(t) = 0$ means that the firm has sold out. Because the seller begins offering single units at time u , it follows that $X_S(0) = aX_B(u)$, which is the remaining inventory after selling bundles for u units of time.

The death process for bundles, $X_B(t)$, has customer arrival rates $\mu_n^B(t) = n\mu_B(t)$, $n = k/a, k/a - 1, \dots, 1, 0$, where n is the number of bundles remaining for sale and $\mu_B(t)$ is a time-dependent death rate. Similarly, the death process for single units, $X_S(t)$, has customer arrival rates of $\mu_n^S(t) = n\mu_S(t)$, $n = j, j - 1, \dots, 1, 0$, where n is the number of individual units remaining, j is the remaining inventory level when the seller puts individual units on sale, and $\mu_S(t)$ is a time-dependent death rate. The death rates, $\mu_B(t)$ and $\mu_S(t)$, are continuously differentiable functions over $[0, T]$ and $\mu_B(t)$ is non-increasing in t . We also assume that the death rate of bundles decreases faster than the rate of single tickets, i.e. $\mu'_B(t) \leq \mu'_S(t)$ for all $t \in [0, T]$. Since the population of potential customers for bundles is smaller than for individual tickets, it is reasonable that the death rate for bundles decrease at least as fast as that for single tickets. Table 1 summarizes the notation employed for the general model.

It follows that $\mu_0^B(\cdot) = \mu_0^S(\cdot) = 0$, which implies that no customers arrive when there is no remaining inventory. This is a necessary condition to ensure that the death processes terminate at state 0 (i.e., with zero units remaining). This assumption may seem to be restrictive, but the non-homogeneous death rate allows us to meet this condition easily. For example, if customer demand for an event were much bigger than the supply, the death rate could adjust sharply to zero only within a small neighborhood of the end of the sales horizon. Similarly, if the demand was exhausted long before the inventory sold out, the death rate could be set to zero when the demand drops off and could remain there until the end of the horizon. Since the demand rate is not fixed over the entire horizon, this model maintains significant degrees of freedom while still ensuring that the demand rate equals zero when inventory is zero. This also aids in the tractability of the model and enables us to identify insights by examining the structure of the optimal decisions.

We assume that customer arrivals in these two demand processes are independent of each other. This coincides with the comments made by an executive with an NBA team that has concluded that the markets for season tickets and single-event tickets are independent from each other. Season tickets appeal to a particular segment of sports fans, while a great many people prefer to purchase single game tickets on an ad hoc basis, often at the box office on the day of the game. There are many reasons for this: busy schedules, weather concerns, out-of-town fans, etc. For one reason or another, these customers are unlikely to make a commitment for more than one game at a time regardless of the team's marketing campaigns and price reductions. Of course, it would be

interesting to expand the model presented here to allow for inter-related demand processes. The dependency of demand on time can be used as a proxy for customer substitution by having the demand rate for single tickets decrease with time.

We also assume that demand is independent of price. The fact that the demand is independent of price simply means that the prices for various tickets have already been established by the organization when they are making this timing decision. Since the ticket prices are fixed in advance, it would be possible to repeat this model for a different set of prices, but we do not conduct this analysis in this study.

Additionally, only one of the demand processes in our model is active at a given time. Only bundles are sold during the time interval $[0, u)$; and once the decision is made to sell single units, they are sold exclusively on $[u, T]$. The first interval corresponds exactly to the way sports and entertainment organizations offer tickets—only season tickets are available at first. When venues put individual tickets on sale, they will still sell season tickets and larger packages; but the vast majority of transactions and the bulk of the marketing efforts are for single-event tickets. Consequently, it is reasonable to assume that only individual units are sold for the latter part of the interval; this assumption coincides with the demand pattern represented in the data set analyzed in Section 5 and other data sets that we have examined.

It is common practice for organizations to begin selling single tickets before the season begins and to put the sales of all tickets on sale at once. Further, the sales of season tickets after singles are on sale or after the beginning of the season is not significant in the groups with whom we worked. It is this initial sales of singles that interest us, thus we focus on the switch from bundle sales to single ticket sales before the performance season begins. This simply means that organizations focus on maximizing the revenue received before the performance season starts. As written, the model assumes no values for leftover goods at the end of the time horizon, although it can also be extended to include a salvage value per ticket that is less than r_S .

3.2 Optimal Timing Decision for Selling Individual Units

Given the above description of the seller’s decision environment, we can characterize her expected revenue function as follows. Let $J(k, T|u)$ denote the seller’s expected revenue given that she has a selling season of length T , a fixed inventory level of k , and she decides to sell large bundles for the time interval $[0, u)$ and individual units over the interval $[u, T]$. The seller faces the following

expected revenue maximization problem:

$$\max_{u \in [0, T]} J(k, T | u) \equiv r_B a E_B[k/a - X_B(u)] + r_S E_S[a E_B[X_B(u)] - X_S(T - u)]. \quad (1)$$

The first part of (1) is the expected revenue obtained from selling bundled units, where $k/a - X_B(u)$ represents the number of bundles sold. Note that the B subscript denotes that the expectation is taken with respect to the death process for bundles. The second part is the expected revenue from the single unit sales. The non-homogeneous death processes with linear death rates result in the state probabilities given in the following theorem. (All proofs and notes are provided in the Appendix.)

Theorem 1 *The state probabilities for the non-homogeneous death process with linear death rate are given by*

$$P_{k/a, n}^B(0, u) \equiv P(X_B(u) = n) = \frac{(k/a)!}{n!(k/a - n)!} q_B^n (1 - q_B)^{k/a - n},$$

where $q_B = e^{-\rho_B(u)}$ and $\rho_B(u) = \int_0^u \mu_B(t') dt'$,

and

$$P_{an, \ell}^S(u, T) \equiv P(X_S(T - u) = \ell) = \frac{(an)!}{\ell!(an - \ell)!} q_S^\ell (1 - q_S)^{an - \ell},$$

where $q_S = e^{-\rho_S(u)}$ and $\rho_S(u) = \int_u^T \mu_S(t') dt'$.

Substituting these probabilities into the seller's expected revenue maximization problem in (1) yields the reduced-form expected revenue function developed in the following lemma.

Lemma 1 *If the demand arrives according to two non-homogeneous pure death processes with linear death rates, the seller's expected revenue function in (1) reduces to*

$$J(k, T | u) = k[r_B(1 - e^{-\rho_B(u)}) + r_S e^{-\rho_B(u)}(1 - e^{-\rho_S(u)})], \quad (2)$$

which is concave in u when $r_B \geq r_S$.

Note that Lemma 1 applies when $r_B \geq r_S$, which follows from the discussion in Section 3.1 about the importance sports and entertainment organizations place on selling ticket packages. The concavity of the expected revenue function in (2) enables us to determine the optimal time at which the seller should begin selling individual tickets, u^* , as indicated in the following theorem.

Theorem 2 *The optimal time, u^* , for the seller to offer single units in the time interval $[0, T]$ when $\mu_S(t) > \mu_B(t)$ for all $t \in [0, T]$ and the assumption in Lemma 1 is satisfied is given by:*

- $u^* = T$, or only bundles are sold, when $(r_B - r_S)\mu_B(T) > (\mu_S(T) - \mu_B(T))r_S$;
- $u^* = 0$, or only single tickets are sold, when $(\mu_S(0) - \mu_B(0))r_S e^{-\rho_S(0)} > (r_B - r_S)\mu_B(0)$;
- the solution of Equation (3) presented below, otherwise:

$$\ln \frac{(r_B - r_S)}{r_S} = \ln \frac{[\mu_S(u) - \mu_B(u)]}{\mu_B(u)} - \rho_S(u). \quad (3)$$

Note that in some cases $u^* = 0$, implying that only individual tickets should be sold. Likewise, $u^* = T$ suggests that the tickets should only be offered as bundles. It is interesting to compare this to what is found in practice. There are many examples of firms practicing mixed bundling, as identified earlier. There are also firms selling season tickets almost exclusively; for example, many popular sports teams such as baseball's Boston Red Sox (2006) and football's Pittsburgh Steelers (2006) offer very few, if any, individual tickets because they have the fan following to fill their stadiums with season-ticket holders without any problems. Examples of firms selling only single tickets are rarer. Some organizations such as *Cirque du Soleil* (2006) and many Broadway theaters that present multiple performances of a single show in a fixed locale only offer individual tickets; there are some customers who see performance multiple times, but these are rare, so the total demand for multiple performances is likely negligible.⁴ Examining the parameter conditions provides insight into the cases where a singular offering is preferable.

The conditions in Theorem 2 that are required for selling only bundles or single tickets throughout the selling season originate from two terms: $(r_B - r_S)\mu_B(t)$ and $(\mu_S(t) - \mu_B(t))r_S e^{-\rho_S(t)}$. The value $e^{-\rho_S(t)}$ can be thought of as the probability that each unit will not be sold even if the goods are sold individually from time t to the end of the selling period. We can interpret $(r_B - r_S)\mu_B(t)$ as the *increased revenue rate* for bundle sales that is gained by delaying the switch further at time t and $(\mu_S(t) - \mu_B(t))r_S e^{-\rho_S(t)}$ as the *possible revenue rate loss* by switching immediately at time t .

The increased revenue rate for the bundle sales is non-increasing in t because $\mu_B(t)$ is non-increasing in t , and the possible revenue rate loss by switching immediately is non-decreasing in t because of the assumptions on death rates in Theorem 2. Therefore, the condition in Theorem 2 that induces the firm to sell only bundles throughout the selling season can be interpreted such that the minimum gain from delaying the switch is larger than the maximum loss from switching immediately. Thus, the gain from delaying the switch at all times $t \in [0, T]$ is bigger than the loss

⁴These cultural organizations often turn to other methods of revenue management such as bundling their performances with travel, lodging, meals, and gambling.

from switching immediately at any t ; therefore $u^* = T$, and only bundles should be offered for the entire selling season.

Similarly, the condition that ensures switching to singles at time zero can be interpreted as having the maximum gain from delaying the switch further being smaller than the smallest possible loss from the switch. Looking at this last condition may give insight on why there are few organizations that sell only single tickets. The right-hand side of the condition in part measures the marginal benefit of a ticket sold in a bundle at rate r_B over a ticket sold individually at rate r_S . If the benefit from bundle-ticket customers is high (including both the nominal price and the intangible benefits discussed earlier), then this condition will not be satisfied. Further, organizations control their selling season horizon, so if they think even a small number of customers may be willing to buy bundles of tickets, then they could make the selling season start earlier to incorporate those customers.

The following corollary establishes that the seller should only offer bundles when the arrival rate and unit revenue for bundles is larger than that for single tickets. This makes sense because not only is the revenue higher for bundles, but the demand rate is higher as well.

Corollary 1 *The optimal time, u^* , for the seller to offer single units in the time interval $[0, T]$ when $r_B \geq r_S$ and $\mu_B(t) \geq \mu_S(t)$ for all $t \in [0, T]$ is given by $u^* = T$. That is, only bundles should be offered for the entire selling horizon.*

3.3 Special Case: Constant Death Rates

The general model presented in Section 3.2 provides a decision methodology that can accommodate a wide variety of demand patterns. The only drawback is that the versatility of this model complicates the form of the optimal solution such that it is difficult to develop any comparative statics for the optimal decision. In this section we consider the simpler case in which the death rates are constant over time; that is, $\mu_B(t) = \mu_B$ and $\mu_S(t) = \mu_S$. It is important to note that this model also has the restrictive property that the demand rate goes to zero linearly as the remaining inventory goes to zero, but it does not afford the flexibility to adjust the demand rate with a piecewise-linear function as the general model allows.

The optimal solution developed in Theorem 2 when $r_B \geq r_S$ and $\mu_B < \mu_S$ now has the following closed form:

$$u^* = \left(T - \frac{\left[\ln \left(\frac{r_S}{r_B - r_S} \right) \left(\frac{\mu_S - \mu_B}{\mu_B} \right) \right]^+}{\mu_S} \right)^+ . \quad (4)$$

We can establish some comparative statics for the optimal timing decision given in (4). While the bundle size, a , does not appear explicitly in the optimal value of u^* , it is reasonable to expect the bundle sales rate, μ_B , to be a function of the bundle size. Suppose that $\mu_B = \gamma/a$, which implies that the demand rate decreases as the bundle size increases. The following lemma describes the relationship between the bundle size and the optimal u^* .

Lemma 2 *Given that $\mu_B = \gamma/a$, as the bundle size increases, the individual tickets should be offered earlier in the selling horizon (i.e., u^* decreases in a).*

Because an increase in the bundle size is assumed to generate a decrease in the bundle demand rate parameter, μ_B , Lemma 2 implies that the optimal u^* value decreases (increases) as μ_B decreases (increases). This is the behavior we would expect because a decrease in μ_B means that selling bundles is less attractive for the seller, so she should find it beneficial to offer them for a shorter period of time.

Examining the behavior of the u^* function as the marginal revenue parameters vary, we can establish the following lemma:

Lemma 3 *As the marginal revenue parameter r_B (r_S) increases, the individual tickets should be offered later (earlier) in the selling horizon.*

The impact of a change in the individual ticket demand rate on the optimal timing decision is a function of the model parameters as detailed in the following lemma.

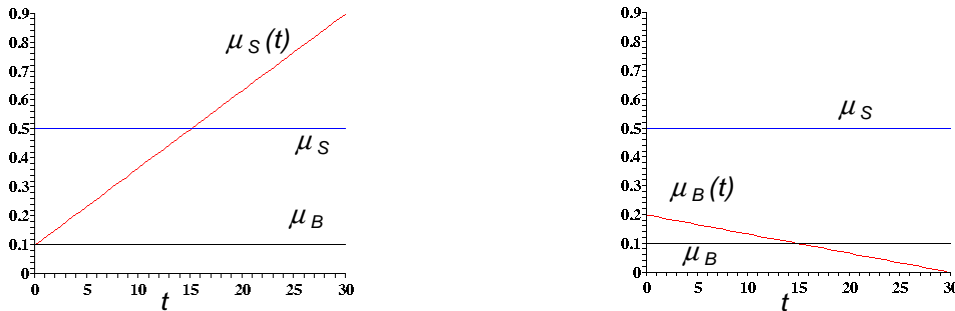
Lemma 4 *As the single ticket demand rate, μ_S , increases, the individual tickets should be offered earlier if $\frac{1}{\mu_S - \mu_B} > \frac{\ln\left[\left(\frac{r_S}{r_B - r_S}\right)\left(\frac{\mu_S - \mu_B}{\mu_B}\right)\right]}{\mu_S}$. Otherwise, individual tickets should be offered later in the time horizon.*

The right-hand side of the condition defined in Lemma 4 represents the amount of time at optimality during which single tickets should be sold in an environment of mixed bundling (as defined in Equation (4)). If the arrival rates are close together, then the percentage increase in the difference between arrival rates is large. Thus, individual tickets are significantly more important than they were before, so they should be offered earlier. If the arrival rates are far apart, an increase in the arrival rate for single tickets increases the chances that remaining tickets will be sold individually when bundles are offered for a longer period of time.

The final comparative static for the optimal timing decision is the interesting observation that the optimal value for u^* is *independent* from the initial inventory level, k . This observation holds

for the general model in Section 3.2 as well as the special case of constant arrival rates. The independence stems from the fact that each state probability has a binomial form, signifying that each bundle and/or unit has the same independent probability of being sold in the time horizon. The seller, therefore, seeks to find the time at which the marginal value of selling a unit in a bundle is equal to its marginal individual sale value, and this time is the same for all initial inventory levels.

While this result runs contrary to our intuition that the initial inventory of goods available for sale should have some impact on the timing decision, the time-dependent arrival rates in the general model can be adjusted with respect to the total amount of inventory. We provide the following numerical example to illustrate how the optimal timing decision changes when arrival rates vary with time.



(a) Increasing Rates for Singles

(b) Decreasing Rates for Bundles

Figure 1: Time dependent rates when $\mu_B = 0.1$, $\mu_S = 0.5$

The base case for this example is when both of the rates for singles and bundles are constant $\mu_B = 0.1$, $\mu_S = 0.5$. We will compare this base rate with two other cases:

1. the death rate for singles is linearly increasing with time from 0.1 to 0.9 (depicted in Figure 1(a)) instead of the constant rate of $\mu_S = 0.5$ (which is equal to the average rate), and
2. the death rate for bundles is linearly decreasing with time from 0.2 to 0 (depicted in Figure 1(b)) instead of the constant rate of $\mu_B = 0.1$ (which is equal to the average rate).

As seen in Figure 2, the fact that the death rates for the bundles are decreasing with time shifts the introduction of single tickets to an earlier time, while the increasing rates for the single tickets delays the switching time.

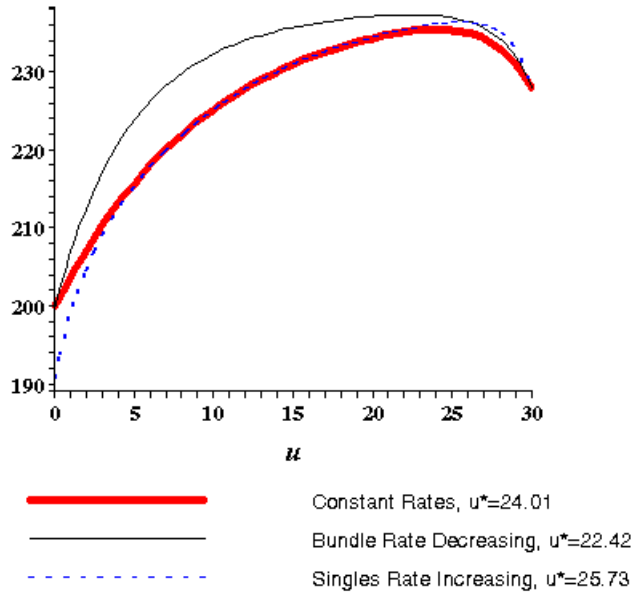


Figure 2: Expected Revenue when $k = 20$, $a = 2$, $T = 30$, $r_S = 10$, $\mu_B = 0.1$, $\mu_S = 0.5$

4 Optimal Timing Decision for Events with Heterogeneous Demand Characteristics

Many sports and entertainment organizations have some events during the performance season that are very popular, and they have no problem selling tickets to these events. They also have a group of events whose tickets are much more difficult to sell for a variety of reasons (e.g., date and time of the event, popularity of the artistic work or opponent, etc.). Another impetus for these organizations to offer ticket bundles is to use the high demand events to induce customers to commit to the other events as well. Thus, in this section we extend the general model developed in Section 3.2 to the situation where the organization sells tickets for events with different demand characteristics.

Let us consider a selling season comprising two distinct events that will take place in a venue with capacity K . One of the events has favorable demand characteristics (called the “high demand event”), and the other one is a less desirable event (“low demand event”). The organization starts the selling season by offering a bundle priced at r_B comprised of one ticket to each event. At time u , it switches to selling high and low demand events individually for the prices of r_H and r_L , respectively. As before, the goal is to determine the optimal time u that maximizes the organization’s revenue over the selling horizon. Although considering a two-event season is a

simplification, it allows us to study the effect of different demand characteristics of events on the optimal switching time; the model could also be extended to more than two performances.

Let $\{X_B(t) : t \in [0, T]\}$, $\{X_L(t) : t \in [0, T]\}$ and $\{X_H(t) : t \in [0, T]\}$ be pure linear death processes with rates $n\mu_B(t)$, $n\mu_L(t)$ and $n\mu_H(t)$, respectively, when the process is in the state (n, t) . Stochastic processes $X_B(t)$, $X_L(t)$, $X_H(t)$ represent the remaining inventory for bundles, the low-demand event, and the high-demand event, respectively. Therefore, we can obtain the probability of selling $(K - n)$ bundles from time zero to u and $(n - \ell)$ individual high and low-demand event tickets from u to T as follows:

$$P_{K,n}^B(0, u) = P(X_B(u) = n) = \frac{K!}{n!(K-n)!} q_B^n (1 - q_B)^{K-n},$$

$$\text{where } q_B = e^{-\rho_B(u)} \text{ and } \rho_B(u) = \int_0^u \mu_B(t') dt';$$

$$P_{n,\ell}^L(u, T) = P(X_L(T - u) = \ell) = \frac{n!}{\ell!(n-\ell)!} q_L^\ell (1 - q_L)^{n-\ell},$$

$$\text{where } q_L = e^{-\rho_L(u)} \text{ and } \rho_L(u) = \int_u^T \mu_L(t') dt';$$

$$P_{n,\ell}^H(u, T) = P(X_H(T - u) = \ell) = \frac{n!}{\ell!(n-\ell)!} q_H^\ell (1 - q_H)^{n-\ell},$$

$$\text{where } q_H = e^{-\rho_H(u)} \text{ and } \rho_H(u) = \int_u^T \mu_H(t') dt'.$$

As in the general model, we assume $\mu_B(t)$, $\mu_L(t)$ and $\mu_H(t)$ are continuously differentiable functions over $[0, T]$, and $\mu_B(t)$ is non-increasing in t . We also assume that the death rate of bundles decreases faster than the rates of high and low-demand event tickets, i.e., $\mu'_B(t) \leq \mu'_L(t)$ and $\mu'_B(t) \leq \mu'_H(t)$ for all $t \in [0, T]$. The seller faces the following maximization problem:

$$\begin{aligned} \max_{u \in [0, T]} J(K, T|u) &\equiv r_B E_B[K - X_B(u)] + r_L E_L[E_B[X_B(u)] - X_L(T - u)] \\ &+ r_H E_H[E_B[X_B(u)] - X_H(T - u)]. \end{aligned} \quad (5)$$

Using the special structure of the non-homogeneous pure death process with linear death rate, we can reduce the expected revenue function in (5) to the following simpler form.

Lemma 5 *The seller's expected revenue function in (5) reduces to*

$$J(K, T|u) = K[r_B(1 - e^{-\rho_B(u)}) + r_L e^{-\rho_B(u)}(1 - e^{-\rho_L(u)}) + r_H e^{-\rho_B(u)}(1 - e^{-\rho_H(u)})], \quad (6)$$

which is concave in u when $r_B \geq r_L + r_H$. Moreover, the optimal value of u is obtained from

$$\mu_B(u)(r_B - r_L - r_H) + r_L[\mu_B(u) - \mu_L(u)]e^{-\rho_L(u)} + r_H[\mu_B(u) - \mu_H(u)]e^{-\rho_H(u)} = 0. \quad (7)$$

Clearly, the value of u^* can be calculated numerically from Equation (7) above, but a closed-form expression for u^* is not obtainable in general. In the rest of this section, we examine a specific structure for modeling the death rates of the single events, which enables us to comment explicitly on the behavior of the optimal switching time.

Let us consider the case when the death rates for the single events are the same for time zero to time T/b ($b > 1$), and after that time the death rate of the low-demand ticket is zero. In this case, demand for the low ticket may drop to zero as the seat quality reduces, where time approximates the remaining inventory. Formally, we have

$$\begin{aligned} \mu_H(t) &= \mu(t) & \text{if } t \in [0, T], \\ \mu_L(t) &= \begin{cases} \mu(t) & \text{if } t \in [0, \frac{T}{b}], \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Then the expected revenue function in Equation (6) becomes

$$J(K, T|u) = \begin{cases} K[r_B(1 - e^{-\rho_B(u)}) + r_L e^{-\rho_B(u)}(1 - e^{-\rho_L(u)}) + r_H e^{-\rho_B(u)}(1 - e^{-\rho_H(u)})] & \text{if } u \in [0, \frac{T}{b}], \\ K[r_B(1 - e^{-\rho_B(u)}) + r_H e^{-\rho_B(u)}(1 - e^{-\rho_H(u)})] & \text{if } u \in [\frac{T}{b}, T]. \end{cases}$$

This expected revenue function is piecewise-concave from Lemmas 1 and 5. Because the expected revenue function is piecewise-concave, we can determine the optimal time to begin selling individual tickets by optimizing each segment of the function individually and then choosing the time that achieves the highest overall expected revenue.

Theorem 3 *The optimal value u^* for the seller to begin offering single tickets in the time interval $[0, T]$ when $\mu(t) > \mu_B(t)$ for all $t \in [0, T]$ and the assumption in Lemma 5 is satisfied is given by:*

$$u^* = \arg \max \{J(K, T | u_1^*), J(K, T | u_2^*)\},$$

where u_1^* is defined by

- $u_1^* = T/b$ when $(r_B - r_L - r_H)\mu_B(T/b) > (\mu(T/b) - \mu_B(T/b))(r_L + r_H e^{-\rho_H(T/b)})$,
- $u_1^* = 0$ when $(\mu(0) - \mu_B(0))(r_L + r_H e^{-\rho_H(T/b)})e^{-\rho_L(0)} > (r_B - r_L - r_H)\mu_B(0)$, and
- the solution of the following equation for $u \in [0, T/b]$ otherwise

$$\ln \frac{(r_B - r_L - r_H)}{r_L + r_H e^{-\rho_H(T/b)}} = \ln \frac{\mu(u) - \mu_B(u)}{\mu_B(u)} - \rho_L(u); \quad (8)$$

u_2^* is defined by

- $u_2^* = T$ when $(r_B - r_H)\mu_B(T) > (\mu(T) - \mu_B(T))r_H$,

- $u_2^* = T/b$ when $(\mu(T/b) - \mu_B(T/b))r_H e^{-\rho_H(T/b)} > (r_B - r_H)\mu_B(T/b)$,
- the solution of the following equation for $u \in [T/b, T]$ otherwise

$$\ln \frac{(r_B - r_H)}{r_H} = \ln \frac{[\mu(u) - \mu_B(u)]}{\mu_B(u)} - \rho_H(u);$$

and $\arg \max f(x)$ is the value of x for which $f(x)$ has the largest value.

It is interesting to note the similarity between the solution in Theorem 3 for events with different demand characteristics to that for the general model with homogeneous events given in Theorem 2. As before, the value $e^{-\rho_L(t)}$ can be thought of as the probability that a ticket to a low-demand event will not be sold even if it is sold individually from time t to T . Similarly, $e^{-\rho_H(t)}$ is the probability that a high-demand ticket will not be sold from time t to T . Also note that because the death rates for the single events are the same for time zero to time T/b , we have $e^{-\rho_H(t)} = e^{-\rho_H(t)}e^{-\rho_H(T/b)} = e^{-\rho_L(t)}e^{-\rho_H(T/b)}$ when $t \leq T/b$.

The conditions in Theorem 3 that are required for switching at the boundary points of the interval $[0, T/b]$ for the profit function $J(K, T | u_1)$ originate from two terms: $(r_B - r_L - r_H)\mu_B(t)$ and $(\mu(t) - \mu_B(t))(r_L e^{-\rho_L(t)} + r_H e^{-\rho_H(t)})$. We can interpret $(r_B - r_L - r_H)\mu_B(t)$ as the increased revenue rate from the bundle sales by delaying the switch further at time t and $(\mu(t) - \mu_B(t))(r_L e^{-\rho_L(t)} + r_H e^{-\rho_H(t)})$ as the possible revenue rate of loss by switching immediately at time t . The increased revenue rate from the bundle sales is non-increasing in t because $\mu_B(t)$ is non-increasing in t , and the possible revenue rate of loss by switching immediately is non-decreasing in t when assumptions for Theorem 3 hold. The interpretation of the conditions in Theorem 3 are similar to the ones in Theorem 2. Moreover, it should be noted that when the switch time is after time T/b , the conditions for u_2^* are identical to the conditions in Theorem 2 by considering the single event as the high-demand event and the selling period as $[T/b, T]$.

Let us examine a numerical example where the death rate of the low-demand event drops to zero before the end of the horizon. Consider a team that faces a two-event selling season, which lasts for 20 units of time. Both of the events will take place in a venue that has 100 seats. The team receives a marginal benefit of \$20 per bundles composed of one high and one low-demand event ticket and charges \$9 for the popular and \$6 for the unpopular event tickets when they are sold separately. Another example considers the results when the team charges \$8 for each single ticket, regardless of the demand characteristics of the events. The death rate parameter for the bundles is $\mu_B(t) = 0.1$ for all $t \in [0, 20]$. When the team sells tickets separately, the death rates are higher, $\mu(t) = 1$, because the customers do not have to commit themselves to two tickets.

The high-demand event's death rate stays constant at 1 throughout the selling season, whereas the death rate of the low-demand event drops to zero after time $20/b$. We consider the following three possible scenarios with respect to different b values and ticket prices.

1. $b = 1$, which means that the death rates for both of the single events are the same. The valuation of the events by customers are different; therefore, ticket prices differ.
2. $b = 2$, which means that the death rate for the low-demand event drops to zero after time 10. The valuation of the events by customers are different; therefore, ticket prices differ.
3. $b = 2$, which means that the death rate for the low-demand event drops to zero after time 10. Customers are charged the same price (\$8) for high and low-demand events when the tickets are sold separately.

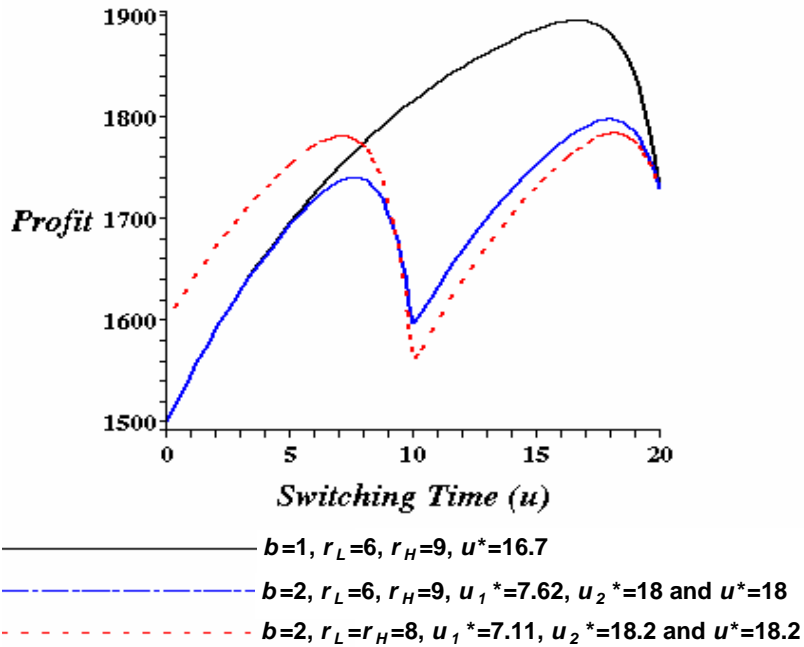


Figure 3: Expected Revenue with different b and single ticket prices when $\mu_B = 0.1$

Figure 3 shows the seller's profit as a function of the switching time to singles. When $\mu_B = 0.1$ for all $t \in [0, T]$, the fact that bundles will sell during the entire horizon with a sufficient demand rate induces the seller to select a late time at which to put individual tickets on sale. The seller takes advantage of the higher bundle price for a long time, and near the end of the horizon she switches

to the singles. At this point customers arrive for high-demand events only in cases 2 and 3 (because no customers want low-demand tickets anymore). The seller was willing to forgo the possibility of selling individual tickets for the low demand event before demand dropped off, because she was able to sell those low-demand tickets in bundles. As seen in Figure 3, the fact that the death rate for the low-demand event is decreasing to zero before the end of the selling horizon in cases 2 and 3 changes the switch time significantly (delays the switch time to 18 as compared to $u^* = 16.7$ in base case 1). It is important to note the implication of the piecewise-concave profit function. The seller could switch early (before $u = 10$) and sell both types of individual tickets, or she could switch later in the horizon and only sell individual tickets for the high demand event.

Figure 4 considers the same three cases with a lower bundle demand rate. Intuition tells us that reducing the bundle demand rate should pull the switch time earlier, making it desirable for the seller to offer single tickets earlier in the selling horizon. When the bundle demand rate is sufficiently small compared to the rates of the single events, the seller may even wish to switch early enough from selling bundles to reach single customers for both the low and high-demand events. Figure 4 shows that when $\mu_B = 0.05$ for all $t \in [0, T]$, the switching times occur dramatically earlier in the selling period in comparison to the base case of $u^* = 15.96$. Moreover, for both of the cases 2 and 3 with the bundle rate of 0.05, the demand rates for both of the single events are still positive when the switch takes place. When comparing Figures 3 and 4, Figure 3 shows that a higher bundle rate (e.g., one that may be driven by the demand for high tickets) moves the switch time later in the time horizon, because more people may commit to the full package.

5 Empirical Analysis

Our main objectives in using empirical data were to verify the plausibility of the demand assumptions we made and also to determine the optimal timing policy for a specific set of data. We examined the history of individual customer ticket sales transactions for the eight months preceding the 2003 Georgia Tech home football season. Georgia Tech's Bobby Dodd Stadium has a capacity of 55,000 seats. The average attendance at home games during the 2003 football season was 48,813, but this includes an average of 7,959 student tickets per game, which were distributed free of charge, as well as the opposing team's allowance, which were not under the control of the Georgia Tech Athletic Association.

Figure 5 depicts the sale of season tickets (six games) and individual tickets per week. In this analysis, we focus on the sales of season tickets and single tickets, although mini-packages were also

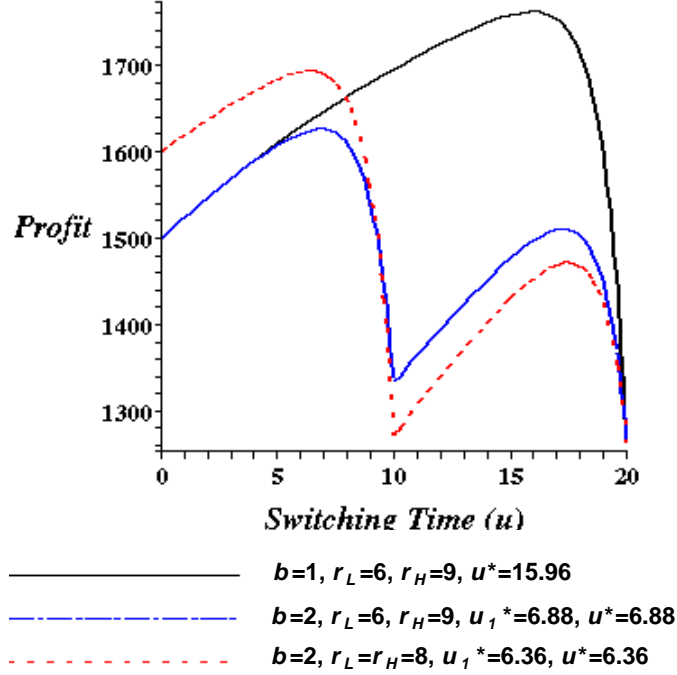


Figure 4: Expected Revenue with different b and single ticket prices when $\mu_B = 0.05$

offered that season. Note that the peaks in demand correspond to the timing of the university's 2003 ticket promotions. We are most interested in the sales pattern following the promotional peak of a ticket product but before the promotion of the next product, because this is likely to represent the natural demand rate as a function of the remaining inventory. It is also important to note that season ticket holders were allowed to buy extra individual tickets at the same time they bought season packages, but the individual tickets were not available to the public at that time. Thus we focus on weeks 0–22 of the season packages and weeks 27–39 of the single ticket sales as they are labelled in the figure.

To estimate the arrival/death rate parameters with respect to time for each type of product, we use the basic relationship that the arrival rate equals $n * \mu(t)$ and use linear regression to estimate the parameter $\mu(t)$ where we know the arrival rate by the sales sold in each week and the corresponding remaining inventory or n_t . A summary of the regression results is provided in Table 2. The first result to notice from the regression result is that the death rate of the bundles is decreasing with time and the rate of its decrease is faster than that of singles, which validates the assumptions made about the demand rates in Section 3.1.

Because in 2003 the team offered all of its packages at the same unit price per ticket regardless

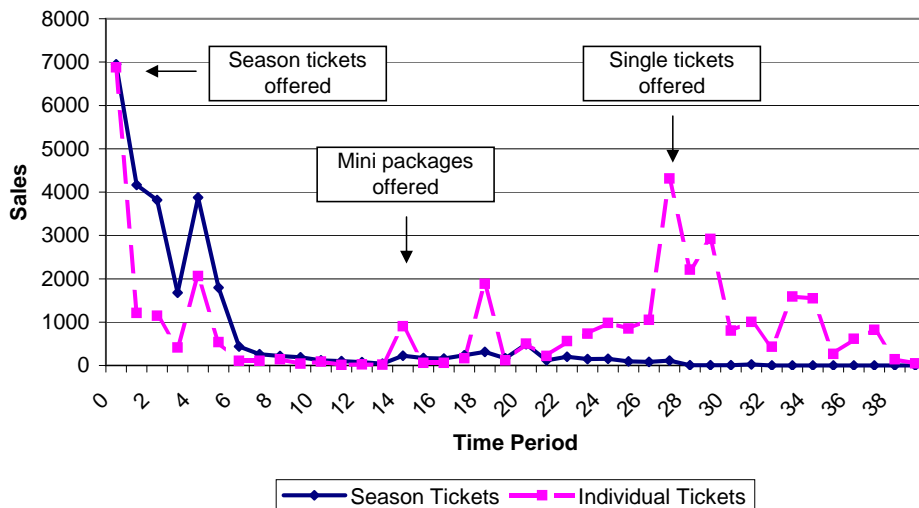


Figure 5: 2003 Georgia Tech Football Ticket Sales of Season Packages and Single Tickets by Week

Tickets	Regression Significance	R^2	Weeks	Intercept Estimate	Slope Estimate
Season Packages	0.00016	0.5185	0-22	0.1307	-0.005352
Single Tickets	0.02751	0.2548	27-39	0.05415	-0.001099

Table 2: Summary of Regression Results

of their size and the demand rate for the bundles are higher than the singles ($\mu_B > \mu_S$) until the 18th week, we can apply Corollary 1 and conclude that a team facing these arrival parameters should offer season tickets as long as possible until week 18. After the 18th week, the rate of singles are bigger ($\mu_S > \mu_B$), and it is easy to see from Theorem 2 that there is no incentive for the team to delay the switch further because team offers all of its packages at the same unit price ($r_B = r_S$). Therefore the optimal switching time for Georgia Tech should be the 18th week.

This is slightly different from Georgia Tech’s 2003 selling policy, which put single tickets on sale with one week left before the season started. We estimate that the team would have benefited by selling more single tickets at the end of the horizon without detracting too much from the season ticket sales; this is approximately when the team offered mini-packages in that selling season. However, we do not have a measure of the intangible benefits of securing season ticket holders; incorporating this estimate into the model would delay the switching time from our current estimates. It is not meaningful for us to quantify the optimality gap in expected revenue between the team’s current policy and the optimal policy because the parameter estimates really depend on the university’s current promotions.

6 Conclusions and Future Research

In this paper we have developed several models for the application of revenue management in the sports and entertainment ticket industry. We studied the sales of season ticket and individual ticket products to customers and the optimal time of switching between product sales. We determined the optimal time at which an organization should start selling individual tickets to events in order to maximize revenue when customers arrive according to two independent, Markovian death processes. We identified the conditions when only bundles or individual tickets should be offered, and we presented some comparative statics for the optimal timing decision under the special case of constant death rates. We also extended the model to the case where the events have different demand characteristics. These models are some of the first to be developed specifically for this increasingly important industry. Hopefully this study will encourage other researchers to consider this industry as well in their work.

It would be interesting in future research to relax some of the assumptions that we included in this model. One of these assumptions was that the demand processes for bundles and single tickets are independent. While these customers are independent in some markets, there undoubtedly exist other markets in which there is significant substitution between the two customer groups. Another possible extension would be to develop a model for risk-averse sellers by explicitly including the seller's discounted cash flow in the expected revenue function. We estimate that this will shift the switch to single tickets to later in the time horizon. See Drake (2006) for an extension of the general model with constant death rates to the environment where the seller offers multiple sizes of bundles along with individual tickets. We also focused on ticket sales in a selling horizon that ended before the performance season began. Some sellers may find it beneficial to stagger the offering of individual tickets to some popular events once the performance season starts to stimulate demand for partial season ticket packages. A final extension would be a new model that considers how sellers could take advantage of this option.

There are many additional research questions outside of the scope of the current model yet to be considered for this industry. Many organizations offer partial-season packages, some of which are comprised of fixed events and some in which the consumers can create their own package. An important research question is how to determine the optimal composition of events for these small packages. The effect of the create-your-own package on the organization's sale of the fixed packages has also not yet been considered. The models presented in this study were developed for the environment of a static timing decision because many organizations currently define this

time before observing actual ticket sales. With the improvement of electronic advertising and sales methods, some organizations may look at delaying this decision in the future; therefore, it would be interesting to consider a dynamic timing decision for this problem as well.

7 Appendix

Proof of Theorem 1. The forward Kolmogorov differential equations for a non-homogeneous death process with linear death rates are given by:

$$\begin{aligned}\frac{\partial p_{m,n}(s,t)}{\partial t} &= \mu(t)(n+1)p_{m,n+1}(s,t) - \mu(t)np_{m,n}(s,t), \quad \forall n > 0, \\ \frac{\partial p_{m,0}(s,t)}{\partial t} &= \mu(t)p_{m,1}(s,t), \quad n = 0.\end{aligned}\tag{9}$$

The transition probability generating function is constructed by multiplying both sides of the equations in (9) by z^n , adding up all equations for $n > 0$ and using the equation for $n = 0$ to eliminate $\frac{\partial p_{m,0}(s,t)}{\partial t}$ from the sum (see Kleinrock (1974) for details of the procedure). Employing this procedure, we obtain

$$\frac{\partial \psi_{m,s}(z,t)}{\partial t} = \mu(t)(1-z)\frac{\partial \psi_{m,s}(z,t)}{\partial z}.$$

In order to solve this differential equation we will use the following theorem from Parzen (1964), which is an application of Lagrange's method for solving first-order linear partial differential equations.

Theorem 4 *Suppose the probability generating function satisfies the partial differential equation*

$$\frac{\partial \psi_{m,s}(z,t)}{\partial t} = a(z,t)\frac{\partial \psi_{m,s}(z,t)}{\partial z}.$$

Let $u(\cdot, \cdot)$ be a function such that the solution $z(t)$ of the differential equation

$$\frac{dz}{dt} + a(z,t) = 0$$

satisfies $u(z,t) = \text{constant}$. Define function $g(\cdot)$ by $g(z) = u(z,s)$, and let $g^{-1}(\cdot)$ be the inverse function of $g(\cdot)$. Then,

$$\psi_{m,s}(z,t) = \{g^{-1}(u(z,t))\}^m.$$

According to the theorem, to obtain the transition probabilities for non-homogeneous linear pure death process, we need to solve the ordinary differential equation,

$$\frac{dz}{dt} + \mu(t)(1-z) = \frac{dz}{(1-z)} + \mu(t)dt = 0.$$

Integrating both sides one obtains

$$\ln(1-z)^{-1} + \int_0^t \mu(t') dt' = c = u(z, t).$$

Let $\rho(s) = \int_0^s \mu(t') dt'$. Then, $u(z, s) = \ln(1-z)^{-1} + \rho(s) = g(z)$. Therefore the inverse of $g(z)$ is given by

$$g^{-1}(\cdot) = 1 - e^{-(x-\rho(s))},$$

and we have the transition generating function as

$$\begin{aligned} \psi_{m,s}(z, t) &= \{g^{-1}(\ln(1-z)^{-1} + \rho(t))\}^m = \{1 - e^{-(\rho(t)-\rho(s))}(1-z)\}^m \\ &= \{1 - e^{-(\rho(t)-\rho(s))} + ze^{-(\rho(t)-\rho(s))}\}^m \end{aligned}$$

Clearly, this is the probability-generating function of a binomial random variable, the number of successes in m trials, with probability $p = e^{-(\rho(t)-\rho(s))}$ of success in each trial. Therefore,

$$p_{m,n}(s, t) = \frac{m!}{n!(m-n)!} e^{-(\rho(t)-\rho(s))n} (1 - e^{-(\rho(t)-\rho(s))})^{m-n}.$$

■

Proof of Lemma 1. The state probabilities given in Theorem 1 are binomial probabilities where the probabilities of a success on a trial are $e^{-\rho_B(u)}$ and $e^{-\rho_S(u)}$ for the bundle ($X_B(u)$) and single-ticket ($X_S(T-u)$) demand processes, respectively, and the number of independent trials for each binomial probability are k/a and an . Therefore, the expected values of these two processes are given by:

$$\sum_{n=0}^{k/a} nP(X_B(u) = n) = \frac{k}{a} e^{-\rho_B(u)} \quad \text{and} \quad \sum_{\ell=0}^{an} \ell P(X_S(T-u) = \ell) = ane^{-\rho_S(u)}.$$

Conditioning first on X_B and then on the process of the single-ticket demand (X_S), we get:

$$J(k, T | u) = kr_B - r_B a \sum_{n=0}^{k/a} nP(X_B(u) = n) + r_S \sum_{n=0}^{k/a} \left(an - \sum_{\ell=0}^{an} \ell P(X_S(T-u) = \ell) \right) P(X_B(u) = n).$$

Substituting the expected values of the binomial processes into $J(k, T | u)$ gives the desired simplification: $J(k, T | u) = kr_B - r_B k e^{-\rho_B(u)} + r_S k (1 - e^{-\rho_S(u)}) e^{-\rho_B(u)}$.

It is easy to see that:

$$\frac{\partial J(k, T | u)}{\partial u} = k\mu_B(u)(r_B - r_S)e^{-\rho_B(u)} + kr_S[\mu_B(u) - \mu_S(u)]e^{-\rho_B(u)-\rho_S(u)}, \quad (10)$$

and

$$\begin{aligned}
\frac{\partial^2 J(k, T|u)}{\partial u^2} &= (\mu'_B(u) - \mu_B^2(u))k(r_B - r_S)e^{-\rho_B(u)} \\
&+ (\mu'_B(u) - \mu'_S(u))e^{-\rho_B(u) - \rho_S(u)}kr_S \\
&- (\mu_B(u) - \mu_S(u))^2e^{-\rho_B(u) - \rho_S(u)}kr_S \\
&\leq 0.
\end{aligned} \tag{11}$$

The last inequality follows from the fact that $r_B \geq r_S$, $e^{-x} > 0 \forall x \geq 0$, $\mu_B(t)$ is non-increasing in t , and $\mu'_B(t) \leq \mu'_S(t)$. This establishes that the expected revenue function in (2) is concave. ■

Proof of Theorem 2. Assuming that $\mu_S(t) > \mu_B(t)$ for all $t \in [0, T]$, the concavity of J established in Lemma 1 allows us to find the optimal value of u by setting the first derivative of J equal to zero.

$$\begin{aligned}
\frac{\partial J(k, T|u)}{\partial u} &= 0 \\
k\mu_B(u)(r_B - r_S)e^{-\rho_B(u)} &= kr_S[\mu_S(u) - \mu_B(u)]e^{-\rho_B(u) - \rho_S(u)} \\
\mu_B(u)(r_B - r_S) &= r_S[\mu_S(u) - \mu_B(u)]e^{-\rho_S(u)} \\
\frac{(r_B - r_S)}{r_S} &= \frac{[\mu_S(u) - \mu_B(u)]}{\mu_B(u)}e^{-\rho_S(u)} \\
\ln \frac{(r_B - r_S)}{r_S} &= \ln \frac{[\mu_S(u) - \mu_B(u)]}{\mu_B(u)} - \rho_S(u)
\end{aligned} \tag{12}$$

The right-hand side of Equation (3) is non-decreasing in u , because

$$\frac{[\mu'_S(u) - \mu'_B(u)]}{[\mu_S(u) - \mu_B(u)]} - \frac{\mu'_B(u)}{\mu_B(u)} \geq 0$$

by the assumptions of Lemma 1 and $\rho_S(u)$ is non-increasing in u . Therefore if the condition:

$$\ln \frac{[\mu_S(0) - \mu_B(0)]}{\mu_B(0)} - \rho_S(0) \leq \ln \frac{(r_B - r_S)}{r_S} \leq \ln \frac{[\mu_S(T) - \mu_B(T)]}{\mu_B(T)}$$

or

$$\frac{\mu_S(0) - \mu_B(0)}{\mu_B(0)}e^{-\rho_S(0)} \leq \frac{r_B - r_S}{r_S} \leq \frac{\mu_S(T) - \mu_B(T)}{\mu_B(T)}$$

is satisfied, the solution u of Equation (3) is in $[0, T]$. If $\frac{r_B - r_S}{r_S} > \frac{\mu_S(T) - \mu_B(T)}{\mu_B(T)}$, then J' is increasing and $u^* = T$, and if $\frac{\mu_S(0) - \mu_B(0)}{\mu_B(0)}e^{-\rho_S(0)} > \frac{(r_B - r_S)}{r_S}$ J' is decreasing and $u^* = 0$. ■

Proof of Corollary 1. When $r_B \geq r_S$ and $\mu_S(t) \leq \mu_B(t)$ for all $t \in [0, T]$, then J' given in (10) is non-decreasing for all values of u ; thus, $u^* = T$. ■

Proof of Lemma 2. We first establish the result for the case where $\mu_S > \mu_B \equiv \gamma/a$. Suppose that $u^* \in (0, T)$ and therefore $u^* = T - \frac{\ln \left[\left(\frac{r_S}{r_B - r_S} \right) \left(\frac{\mu_S - \gamma/a}{\gamma/a} \right) \right]}{\mu_S}$; then by taking the partial derivative

of u^* with respect to a , we obtain $\frac{\partial u^*}{\partial a} = \frac{-1}{a(\mu_S - \gamma/a)}$. This ratio is always negative because $\mu_S > \gamma/a$; thus, u^* is decreasing in a .

The condition for $u^* = 0$ is $\frac{r_B - r_S}{r_S} < e^{-\mu_S T} \left(\frac{\mu_S - \gamma/a}{\gamma/a} \right)$. As a increases, the ratio on the left remains constant while the ratio on the right increases. This means that it is easier for this condition to be satisfied; as a result, the decision remains at $u^* = 0$ as a increases.

The condition for $u^* = T$ is $\frac{r_B - r_S}{r_S} > \frac{\mu_S - \gamma/a}{\gamma/a}$. As a increases, the ratio on the left remains the same while the ratio on the right increases. This implies that the optimal u^* moves away from T because it is more difficult for this relationship to hold as a increases.

If $\mu_S \leq \gamma/a$, then $u^* = T$ as a increases initially and until a is large enough that $\mu_S > \gamma/a$. After that point, the policy transitions to the above case. ■

Proof of Lemma 3. Suppose that $u^* \in (0, T)$; therefore, $u^* = T - \frac{\ln \left[\left(\frac{r_S}{r_B - r_S} \right) \left(\frac{\mu_S - \mu_B}{\mu_B} \right) \right]}{\mu_S}$, and $\frac{\partial u^*}{\partial r_B} = \frac{1}{\mu_S(r_B - r_S)}$. This ratio is always positive because we are still assuming $r_B \geq r_S$; thus, u^* is increasing in r_B . Looking at the conditions under which $u^* = 0$ and $u^* = T$, we see that the optimal decision is moving toward T and away from 0 because the condition for T to be optimal is easier to satisfy as r_B increases.

Similarly, $\frac{\partial u^*}{\partial r_S} = \frac{-r_B}{\mu_S r_S (r_B - r_S)}$, which is always negative, so u^* is decreasing as r_S increases. Examination of the optimal decisions at the endpoints of the horizon again yields the result that the decision moves toward 0 and away from T . ■

Proof of Lemma 4. Suppose that $u^* \in (0, T)$; then $\frac{\partial u^*}{\partial \mu_S} = \frac{-1}{\mu_S(\mu_S - \mu_B)} + \frac{\ln \left[\left(\frac{r_S}{r_B - r_S} \right) \left(\frac{\mu_S - \mu_B}{\mu_B} \right) \right]}{\mu_S^2}$. Setting $\frac{\partial u^*}{\partial \mu_S} < 0$, we obtain the condition under which u^* is decreasing in μ_S . Setting the partial derivative greater than zero establishes the converse result. ■

Proof of Lemma 5. The state probabilities for the processes $X_B(u)$, $X_L(T - u)$, and $X_H(T - u)$ are binomial probabilities where the probabilities of a success on a trial are $e^{-\rho_B(u)}$, $e^{-\rho_L(u)}$, and $e^{-\rho_H(u)}$, respectively, and the number of independent trials for each binomial probability are K for $X_B(u)$ and n for $X_L(T - u)$ and $X_H(T - u)$. Therefore, the expected values of these three processes are given by: $\sum_{n=0}^K n P(X_B(u) = n) = K e^{-\rho_B(u)}$, $\sum_{\ell=0}^n \ell P(X_L(T - u) = \ell) = n e^{-\rho_L(u)}$, and $\sum_{\ell=0}^n \ell P(X_H(T - u) = \ell) = n e^{-\rho_H(u)}$.

Conditioning first on X_B and then on the processes for the two single-ticket demand (X_L and X_H), we get:

$$\begin{aligned} J(K, T | u) &= r_B \sum_{n=0}^K (K - n) P(X_B(u) = n) + r_L \sum_{n=0}^K \left(\sum_{\ell=0}^n (n - \ell) P(X_L(T - u) = \ell) \right) P(X_B(u) = n) \\ &+ r_H \sum_{n=0}^K \left(\sum_{\ell=0}^n (n - \ell) P(X_H(T - u) = \ell) \right) P(X_B(u) = n) \end{aligned}$$

Substituting the expected values of the binomial processes into $J(K, T | u)$ gives the desired simplification:

$$J(K, T | u) = Kr_B(1 - e^{-\rho_B(u)}) + Kr_L(1 - e^{-\rho_L(u)})e^{-\rho_B(u)} + Kr_H(1 - e^{-\rho_H(u)})e^{-\rho_B(u)}.$$

It is easy to see that:

$$\begin{aligned} \frac{\partial J(K, T | u)}{\partial u} &= K\mu_B(u)(r_B - r_L - r_H)e^{-\rho_B(u)} + Kr_L[\mu_B(u) - \mu_L(u)]e^{-\rho_B(u) - \rho_L(u)} \\ &+ Kr_H[\mu_B(u) - \mu_H(u)]e^{-\rho_B(u) - \rho_H(u)}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 J(K, T | u)}{\partial u^2} &= (\mu'_B(u) - \mu_B^2(u))K(r_B - r_L - r_H)e^{-\rho_B(u)} \\ &+ (\mu'_B(u) - \mu'_L(u))e^{-\rho_B(u) - \rho_L(u)}Kr_L \\ &- (\mu_B(u) - \mu_L(u))^2e^{-\rho_B(u) - \rho_L(u)}Kr_L \\ &+ (\mu'_B(u) - \mu'_H(u))e^{-\rho_B(u) - \rho_H(u)}Kr_H \\ &- (\mu_B(u) - \mu_H(u))^2e^{-\rho_B(u) - \rho_H(u)}Kr_H \\ &\leq 0. \end{aligned}$$

The last inequality follows from: $r_B \geq r_L + r_H$, $e^{-x} > 0 \forall x \geq 0$, $\mu_B(t)$ is non-increasing in t , $\mu'_B(t) < \mu'_L(t)$ and $\mu'_B(t) < \mu'_H(t)$. Thus, $J(K, T | u)$ is concave in u . Moreover, the concavity of $J(K, T | u)$ allows us to solve for the optimal value of u by setting $\frac{\partial J(K, T | u)}{\partial u}$ equal to zero. ■

Proof of Theorem 3. Assuming $\mu(t) > \mu_B(t)$ for all $t \in [0, T]$, the concavity of J in the interval $[0, T/b]$ established in Lemma 5 allows us to find the optimal value u_1^* by setting the first derivative of J in interval $[0, T/b]$ equal to zero. Therefore from Equation (7) we have:

$$\begin{aligned} (r_B - r_L - r_H) &= \frac{\mu(u) - \mu_B(u)}{\mu_B(u)}(r_L e^{-\int_u^{T/b} \mu(y) dy} + r_H e^{-\int_u^T \mu(y) dy}) \\ (r_B - r_L - r_H) &= \frac{\mu(u) - \mu_B(u)}{\mu_B(u)} e^{-\int_u^{T/b} \mu(y) dy} (r_L + r_H e^{-\int_{T/b}^T \mu(y) dy}) \\ \ln(r_B - r_L - r_H) &= \ln \frac{\mu(u) - \mu_B(u)}{\mu_B(u)} - \int_u^{T/b} \mu(y) dy + \ln(r_L + r_H e^{-\int_{T/b}^T \mu(y) dy}) \\ \ln \frac{(r_B - r_L - r_H)}{r_L + r_H e^{-\int_{T/b}^T \mu(y) dy}} &= \ln \frac{\mu(u) - \mu_B(u)}{\mu_B(u)} - \int_u^{T/b} \mu(y) dy \\ \ln \frac{(r_B - r_L - r_H)}{r_L + r_H e^{-\rho_H(T/b)}} &= \ln \frac{\mu(u) - \mu_B(u)}{\mu_B(u)} - \rho_L(u). \end{aligned}$$

The last equation above is the same as that given in Equation (8) in Section 4. The right-hand side of Equation (8) is non-decreasing in u , because

$$\frac{[\mu'(u) - \mu'_B(u)]}{[\mu(u) - \mu_B(u)]} - \frac{\mu_B(u)'}{\mu_B(u)} \geq 0$$

by the assumptions about the death rates in Section 4 and because $\rho_L(u)$ is non-increasing in u . Therefore if the condition:

$$\ln \frac{[\mu(0) - \mu_B(0)]}{\mu_B(0)} - \rho_L(0) \leq \ln \frac{(r_B - r_L - r_H)}{r_L + r_H e^{-\rho_H(T/b)}} \leq \ln \frac{[\mu(T/b) - \mu_B(T/b)]}{\mu_B(T/b)}$$

or

$$\frac{\mu(0) - \mu_B(0)}{\mu_B(0)} e^{-\rho_L(0)} \leq \frac{(r_B - r_L - r_H)}{r_L + r_H e^{-\rho_H(T/b)}} \leq \frac{\mu(T/b) - \mu_B(T/b)}{\mu_B(T/b)}$$

is satisfied, the solution u_1^* is in $[0, T/b]$.

If $\frac{(r_B - r_L - r_H)}{r_L + r_H e^{-\rho_H(T/b)}} > \frac{\mu(T/b) - \mu_B(T/b)}{\mu_B(T/b)}$, then J' is increasing in $u \in [0, T]$ and $u_1^* = T/b$. J' is decreasing in $u \in [0, T]$, and therefore $u_1^* = 0$ if $\frac{\mu(0) - \mu_B(0)}{\mu_B(0)} e^{-\rho_L(0)} > \frac{(r_B - r_L - r_H)}{r_L + r_H e^{-\rho_H(T/b)}}$.

The derivation of u_2 is the same as the derivation of u_1 but with r_S replaced by r_H and $[0, T]$ by $[T/b, T]$. ■

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