Dynamic Pricing and the Direct-to-Customer Model in the Automotive Industry: Technical Results and Proofs*

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Abstract

Many manufacturers are looking towards dynamic pricing, especially when combined with the Direct-to-Customer business model to increase profits and improve supply chain performance. To illustrate these benefits, we discuss a strategy that incorporates pricing, production scheduling, and inventory control under production capacity limits in a multi-period horizon. We show that under concave revenue curves, a greedy algorithm provides the optimal solution. This technical report is the companion to the article with the same title, appearing in the E-Commerce Journal in Fall 2002. In this report we present the details of the technical results and proofs outlined in the paper.

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1 Introduction and Literature Review

We focus on the coordination of pricing, production and distribution decisions for non-perishable products in a multi-period time horizon, where planning in advance is a key element. We allow for periodically varying parameters such as demand curves, capacity limits, holding costs and production costs. The individual revenue curves are concave, and the objective is to maximize profit; general properties of the problem allow it to be solved efficiently with the greedy algorithm. Extensions to the model include the addition of production set-up cost and allowing multiple products to share common production capacity. A full description of the model and extensions is provided in Biller et al (2002); in this technical report we present the technical results and proofs from the prior article in detail.

As far as we know, Whitin (1955) is the first to suggest the need to consider joint pricing and inventory control strategies in a non-perishable environment such as retailing. In this paper, Whitin examined a single period problem, most similar to a "newsboy" problem, and determines a single price and supply quantity. Numerous other researchers have considered price determination and restocking in a multi-period setting. For example, Thowsen (1975), and Zabel (1972) both consider multi-period models with a convex ordering cost structure. The retail industry, particularly fashion items with seasonality, has also seen application of price differentiation policies and coordination of inventory control, in some cases under the name "yield management". For instance, Gallego and van Ryzin (1994) analyzed the dynamic adjustment of price as a function of inventory and length of remaining sales; the demand was stochastic but restocking was not allowed. A thorough review of both single and multi-period models combining pricing and inventory strategies can be found in Eliashberg and Steinberg (1991).

In a manufacturing environment, a dynamic pricing model must determine prices as well as inventory levels. However, unlike retail dynamic pricing models, a manufacturing model must also schedule production and account for limitations in production capacity. In the case of multiple products, the strategy must also reflect shared production capacity among products. To date, few models have encompassed all of the necessary requirements.

The most notable exception, however, is the work by Federgruen and Heching (1999), who address the problem of determining optimal pricing and inventory control strategies under demand uncertainty; their model can also be extended to cover capacity limits on production. Indeed, their model is similar to our model, except for two important differences: Federgruen and Heching allow for stochastic demand but require fully backlogged stockouts, i.e., a customer purchasing an item which is out of inventory would receive the product as soon as it becomes available. In contrast, our model is deterministic but allows for lost sales; in addition, extensions to our model include multiple products sharing common production capacity and the addition of production set-up cost.

Some dynamic pricing problems may also be viewed as a special case of resource allocation problems. In these problems, we are given a fixed amount of resources, e.g., production and distribution capacity. Our objective is to allocate the resources to activities, e.g., production and distribution, so as to maximize a certain objective function, e.g., profit. For a comprehensive examination of resource allocation problems and algorithms, see Ibaraki and Katoh (1988).

An important algorithm for resource allocation problems is the greedy algorithm, which is known to be optimal under certain conditions. The greedy approach assigns one unit of resource at each iteration to the activity which contributes most favorably to the objective until the constraint set is tight or no activity is found. This algorithm is also known as a marginal allocation or incremental algorithm.

Chan, Simchi-Levi and Swann (CSS) (2002) showed that a certain class of resource allocation problem can be solved with the greedy algorithm. In particular, they defined a class of functions called *lightly concave*, and considered problems with a polymatroid feasible region, showing that the greedy algorithm provides the optimal solution for problems in this class. In the following sections we will review their result and show that the pricing problem we consider falls in the class of problems that can be solved by the greedy algorithm.

2 Mathematical Model and Technical Results

2.1 Preliminary Notation and Previous Results

In order to present the pertinent results from CSS (2002), we present necessary notation below.

Consider a finite index set $\{1, 2, \ldots, E\}$, also referred to as set E, and let V be a non-negative real function defined on 2^E, i.e. $V : 2^E \to \mathcal{R}$, where $2^E = \{S | S \subseteq E\}$.

Björner and Ziegler (1992) define a polymatroid as follows: A pair (E, V), consisting of a finite ground set E and a function $V : 2^E \to \mathcal{R}$, is called a *polymatroid* if for all $S, T \subseteq E$:

1. $f(\emptyset) = 0$;
2. $S \subseteq T$ implies $V(S) \le V(T)$;
3. $V(S \cap T) + V(S \cup T) \le V(S) + V(T)$.

Let $F = (E, V)$ be a polymatroid and $f(x)$ be a cost function defined on $x \in \mathcal{N}^E$. We focus on the following general resource allocation problem, referred to as Problem $P(f, F)$:

$$P(f, F): \quad max\Big\{f(x) | x \in F\Big\}. \tag{1}$$

We say that $x \in F$ is a *global optimum* of $P(f, F)$ if and only if $f(x) \ge f(y)$ for all $y \in F$.

The main result of CSS (2002) is that the *greedy algorithm* solves $P(f, F)$ under certain assumptions on the cost function $f(\cdot)$ and the constraint set F. This algorithm can be described as follows:

Greedy Algorithm:

Step 0: $x = 0$;

Step 1: Find $i \in E$ with $x + e^i \in F$, $f(x + e^i) \geq f(x)$ and $f(x + e^i) \geq f(x + e^j)$ where $j \in E$ and $x + e^j \in F$

Step 2: If no such $i \in E$ exists, stop.

Step 3: $x = x + e^i$ and go to step 1.

CSS (2002) defined a class of functions called *lightly concave*. A function $f(\cdot)$ is lightly concave with respect to a polymatroid if it satisfies:

(L1) if $y \geq x, f(x) \geq f(x + e^i)$, then $f(y) \geq f(y + e^i), \quad i \in E$
(L2) if $y, x + e^i \in F, y > x, y_i = x_i$ with $f(x + e^i) \geq f(x + e^j)$ for all $x + e^j \in F$, then there exists $y + e^i - e^l \in F$ such that $f(y + e^i - e^l) \geq f(y)$ and $y_l > x_l$.

The main result from CSS (2002) is shown in the following theorem:

Theorem 2.1 (CSS (2002) Main Result) *If a function $f(\cdot)$ is lightly concave with respect to a polymatroid feasible region F, then the greedy algorithm generates an optimal solution for $P(f, F)$, defined in (1).*

We will show in following sections that the pricing problem we consider falls within the class defined by CSS (2002), and thus the greedy algorithm provides an efficient method for solving these problems.

2.2 Mathematical Formulation of the Pricing Problem

Consider a single facility that must determine prices and production scheduling for a single product over a finite horizon. For each period $t = 1, \ldots, T$, let X_t, D_t, and I_t be the amount of product produced, the demand satisfied, and the end of period inventory, respectively.

The facility may produce a maximum of Q_t products in period t, and production cost incurred in period t is k_t per unit produced. Production costs are initially assumed to be linear. Inventory holding cost at a rate of h_t dollars per unit is charged for any inventory carried from period $t - 1$ to t. All cost and capacity parameters may vary from period to period.

We assume that demand is a non-increasing function of the price of the product, and these demand curves may vary from period to period. Thus, by determining the satisfied demand or sales in each period, we simultaneously determine the price of the product in each period. In the model, there is no time lag between a price change and the corresponding change in demand. We also assume that demand occurs in discrete units, and thus so does production. Since in our model we allow for limits on production, it may not be possible to satisfy all observed demand that occurs in a period; the amount of demand that occurred but was not satisfied is lost sales.

We allow for upper and lower limits on price or demand as well. Lower limits on demand ensure a minimum market share of a product, and price bounds may be used to stay within reasonable ranges compared to competitors. In addition, methods for estimating demand curves may provide parameters that are valid only within certain ranges.

The revenue occurring in each period, $R_t(D_t)$, is the selling price times the amount sold, or $P_t * D_t$. It is assumed that the revenue function is a concave function of the sales in each period. Linear demand curves are one example of demand-price functions that satisfy this property. The revenue function also allows for bounds on price or demand in any period t.

The objective of the pricing problem is to maximize total profit over the T periods, considering revenue, holding costs, and production costs in each period. Beginning inventory is zero, and there are production capacity limits in each period. The pricing problem, referred to as Problem PP, can be formulated as:

$$
\begin{array}{lll}
\text{(PP)} & \max & \sum_{t=1}^{T}(R_t(D_t) - h_t I_t - k_t X_t) \\
 & \text{subject to} & I_1 = 0 \\
 & & I_{t+1} = I_t + X_t - D_t, \quad t = 1, 2, \ldots, T \\
 & & X_t \leq Q_t, \quad t = 1, 2, \ldots, T \\
 & & I_t, X_t, D_t \text{ integer} \geq 0, \quad t = 1, 2, \ldots, T.
\end{array}
$$

Observe that in this model, the decision variables are the inventory level at the beginning of the period,I_t, production level X_t, and satisfied demand D_t. Since demand is a non-increasing function of price, demand satisfied in period t, D_t, will uniquely determine the product price, P_t. Our objective is to maximize total revenue minus holding and production costs. The first constraint indicates that there is no inventory at the beginning of period 1. The second set of constraints balances the inventory at each period. The third set of constraints ensures that production capacities are not exceeded.

2.3 Theoretical Results

Problem PP can be described as a min-cost network flow problem with (negative) convex cost and thus standard network flow algorithms for convex cost flows can be applied, see Ahuja, Magnanti and Orlin (1993). However, there are also extensions of the pricing problem which cannot be solved by network flow algorithms, for example, certain multi-product models sharing common components (see Swann 2001).

In this section, we prove the following theorem:

Theorem 2.2 *If the revenue functions are concave in Problem PP, then the objective function $f(\cdot)$ is lightly concave with respect to a polymatroid feasible region F.*

We begin by showing that Problem PP has a polymatroid feasible region, then we show that the objective function is lightly concave as defined in CSS (2002). Of course, the implication is that the greedy algorithm solves pricing problem PP. Furthermore, following the same method, it is possible to show that the multi-product/multi-component problems mentioned above can also be solved by the greedy algorithm.

To show that the feasible region of Problem PP is a polymatroid, we first show it as a network-based model, see Figure 1 for a graphical depiction.

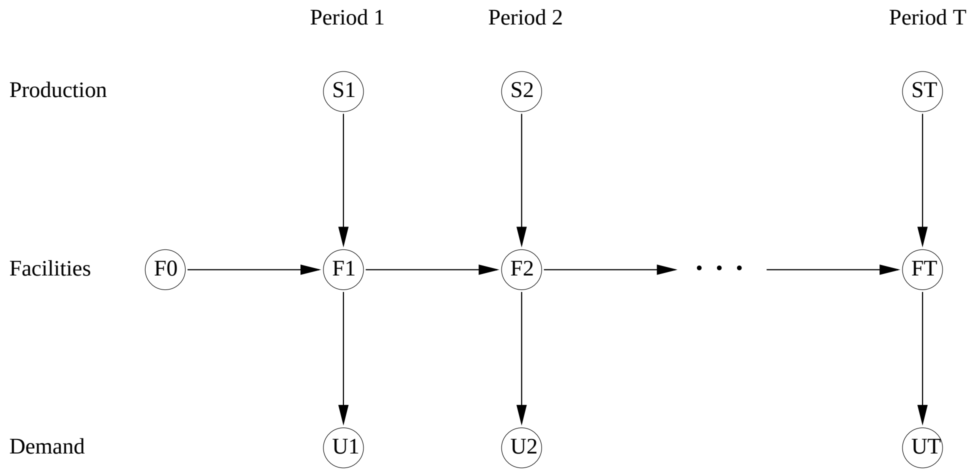

Figure 1: Network Formulation of Problem PP

For each period $t = 1, 2, \ldots, T$, nodes F_t, S_t and U_t represent the facility, production and demand satisfied, respectively. Node F_0 represents the beginning of the planning horizon, i.e., the state of the facility at the beginning of period 1. The directed arcs $(F_{t-1}, F_t), (S_t, F_t)$ and (F_t, U_t) indicate the inventory flow from the previous period, the production flow to the facility and the flow of demand satisfied at period t, respectively. Consider the directed network G with node set $N = \{F_0,\ F_t, S_t, U_t | t = 1, 2, \ldots, T\}$ and arc set $A = \{(F_{t-1}, F_t), (S_t, F_t), (F_t, U_t) | t = 1, 2, \ldots, T\}$, and define the following variables:

x_{ij} = flow on arc $(i, j) \in A$;

D_t = demand satisfied at period $t, t = 1, 2, \ldots, T$.

To describe the feasible region of Problem PP as a network flow model, we impose the following constraints on the network G:

$$
\begin{aligned}
& 0 \le x_{S_t F_t} \le Q_t, t = 1, 2, \ldots, T, \\
& x_{F_t U_t} = D_t, t = 1, 2, \ldots, T, \\
& \sum_{l|(j,l)\in A} x_{jl} - \sum_{l|(l,j)\in A} x_{lj} = 0, \qquad j \in \{F_t | t = 1, 2, \ldots, T\}, \\
& x_{F_0 F_1} = 0, \\
& x_{ij} \ge 0, \text{ and integer } \forall (i, j) \in A, \\
& D_t \ge 0, t = 1, 2, \ldots, T.
\end{aligned} \tag{2}
$$

The first set of constraints ensures that the production capacities are not exceeded, while the second set of constraints guarantees that D_t units of demand are satisfied at each period t. The third set of constraints is a flow conservation constraint, and zero inventory at the beginning of period 1 is guaranteed by the fourth constraint.

Let $D = (D_1, D_2, \ldots, D_T)$ and $x = (x_{ij})_{(i,j)\in A}$. Megiddo (1974) showed that the set

$$\{D \in \mathcal{N}^T | (D, x) \text{ satisfies the constraints in (2) for a given vector } x\}$$

is a polymatroid.

To formulate Problem PP as Problem $P(f, F)$, we need to define a function $f(\cdot)$. Thus, for any given $D \in F \subseteq \mathcal{N}^T$, let $f(D)$ be the optimal objective value of

$$\begin{array}{lll} \text{(PP(D))} & \max_x & \sum_{t=1}^{T}(R_t(D_t) - h_t I_t - k_t X_t) \\ & \text{subject to} & I_1 = 0 \\ & & I_{t+1} = I_t + X_t - D_t, \quad t = 1, 2, \ldots, T \\ & & X_t \leq Q_t, \quad t = 1, 2, \ldots, T \\ & & I_t, X_t, D_t \text{ integer} \geq 0, \quad t = 1, 2, \ldots, T. \end{array}$$

Otherwise, let $f(D) = -1$. Hence, Problem PP is equivalent to Problem $P(f, F)$ with $f(\cdot)$ being the function induced by the real value function f.

To show that the greedy algorithm generates an optimal solution for the pricing problem with concave revenue functions, it remains to prove that the objective function of Problem PP is lightly concave. An outline of the proof and related properties is provided below.

Given a vector D, we first show several properties of an optimal solution for Problem (PP(D)). Notice that for any given D, problem (PP(D)) is equivalent to:

$$\begin{array}{lll} \text{(PP(D}\prime\text{))} & \min & \sum_{i=1}^{T}(k_i X_i + h_i I_i) \\ & \text{subject to} & I_1 = 0 \\ & & I_{t+1} = I_t + X_t - D_t, \quad i = 1, 2, \ldots, T \\ & & X_t \leq Q_t, \quad t = 1, 2, \ldots, T \\ & & I_t, x_t \text{ integer} \geq 0, \quad t = 1, 2, \ldots, T. \end{array}$$

Problem (PP(D′)) is a capacitated inventory model with linear and thus a convex objective function. For this problem, Johnson (1957) proved the following property:

Property 2.3 *For any $D^1 \leq D^2$, there exists a pair of optimal solutions such that $X^1 \leq X^2$.*

It is also useful to observe the following regarding production periods and demand consumption periods.

Observation 2.4 *Consider problem (PP (D′)) given a feasible solution and a specific period t. If demand in period t is increased by one unit, it may be feasible to satisfy this increase in demand by increasing production (if possible) in the given policy at time s by one unit, for some $s > t$.*

Clearly the production period s may be prior to the demand period t if there is available production capacity in s. However, the production period may follow the demand period as well; in this case, the inventory levels between t and s must all be positive. In that case, one unit of product produced no later than t is reallocated to satisfy demand in t, inventory between t and s is decreased by one unit, and production is increased in period s by one unit.

To show that $f(\cdot)$ satisfies lightly concave properties $L1$ and $L3$ within the polymatroid feasible region, we condition on the periods in which to increase demand and production, and further subdivide cases according to the inventory levels between the periods.

2.4 Proof of Theorem 2.2

For problem (PP (D/)), for any D^α, let x^α, I^α and $f(D^\alpha)$ be the corresponding optimal production schedule, inventory and objective function value, respectively.

To prove Theorem 2.2, we need to show that $f(\cdot)$ satisfies the lightly concave properties (L1) and (L2). We begin by showing that the function $f(\cdot)$ satisfies (L1) within the feasible region. To do this, we first show a useful lemma.

Lemma 2.5 *If $D^1 < D^2$ and $D^2 + e^i \in F$, then $f(D^2 + e^i) - f(D^2) \leq f(D^1 + e^i) - f(D^1)$.*

Proof. Let $D^3 = D^2 + e^i$ and $X_j^3 = X_j^2 + 1$ with $X^3 \geq X^2$. Note that i is the period in which demand has increased by one unit, as we move from D^2 to D^3, whereas j is the period in which production has increased by one unit. We divide the proof according to the relative positions of i and j.

- Case i: $j \leq i$, then

$$\begin{aligned} f(D^2 + e^i) - f(D^2) &= R_i(D_i^2 + 1) - R_i(D_i^2) - k_j - \textstyle\sum_{t=j+1}^{i} h_t \\ &\leq R_i(D_i^1 + 1) - R_i(D_i^1) - k_j - \textstyle\sum_{t=j+1}^{i} h_t \\ &\leq f(D^1 + e^i) - f(D^1). \end{aligned}$$

  The last inequality holds since $X^1 + e^j$ is feasible for $D^1 + e^i$.

- Case ii: $j > i$, we further divide the proof into sub-cases according to the inventory levels between periods i and j in I^1.

  - Case ii(a): $I_{t+1}^1 > 0, \quad t = i, i+1, \ldots, j-1$, then

$$\begin{aligned} f(D^2 + e^i) - f(D^2) &= R_i(D_i^2 + 1) - R_i(D_i^2) - k_j + \textstyle\sum_{t=i+1}^{j} h_t \\ &\leq R_i(D_i^1 + 1) - R_i(D_i^1) - k_j + \textstyle\sum_{t=i+1}^{j} h_t \\ &\leq f(D^1 + e^i) - f(D^1), \end{aligned}$$

    where the last inequality is again justified by the fact that $X^1 + e^j$ is feasible for $D^1 + e^i$.

– Case ii(b): $I^1_{t+1} = 0$ for some $t = i, i+1, \dots, j-1$. In this case, we first look for a suitable feasible solution for $D^1 + e^i$. For this purpose, let $l \geq i$ be the smallest index with $I^1_{t+1} = 0$. Using Property 2.3 we can choose solutions such that $X^2 \geq X^1$ and since j is the optimal time to increase production by one unit for D^3 we have $I^2_{l+1} > I^1_{l+1} = 0$. Therefore,

$$\sum_{t=l+1}^{T} D^2_t = I^2_{l+1} + \sum_{t=l+1}^{T} X^2_t > I^1_{l+1} + \sum_{t=l+1}^{T} X^1_t = \sum_{t=l+1}^{T} D^1_t$$

and

$$\sum_{t=1}^{l} X^2_t = \sum_{t=1}^{l} D^2_t + I^2_{l+1} > \sum_{t=1}^{l} D^1_t + I^1_{l+1} = \sum_{t=1}^{l} X^1_t.$$

Observe that these inequalities imply that there exist indices a and b as follows. Let $a > l$ be the smallest index with $D^2_a > D^1_a$ and $b \leq l$ be the largest index with $X^2_b > X^1_b$. Then for $t = l+1, l+2, \dots, a-1$,

$$\begin{aligned} I^2_{t+1} &= I^2_{l+1} + \sum_{\alpha=l+1}^{t} X^2_\alpha - \sum_{\alpha=l+1}^{t} D^2_\alpha \\ &\geq I^2_{l+1} + \sum_{\alpha=l+1}^{t} X^1_\alpha - \sum_{\alpha=l+1}^{t} D^1_\alpha \\ &> I^1_{l+1} + \sum_{\alpha=l+1}^{t} X^1_\alpha - \sum_{\alpha=l+1}^{t} D^1_\alpha \\ &= I^1_{t+1} \geq 0. \end{aligned}$$

For $t = b, b+1, \dots, l-1$,

$$\begin{aligned} I^2_{t+1} &= I^2_{l+1} - \sum_{\alpha=t+1}^{l} X^2_\alpha + \sum_{\alpha=t+1}^{l} D^2_\alpha \\ &\geq I^2_{l+1} - \sum_{\alpha=t+1}^{l} X^1_\alpha + \sum_{\alpha=t+1}^{l} D^1_\alpha \\ &> I^1_{l+1} - \sum_{\alpha=t+1}^{l} X^1_\alpha + \sum_{\alpha=t+1}^{l} D^1_\alpha \\ &= I^1_{t+1} \geq 0. \end{aligned}$$

Hence $X^2 + e^b$ is feasible for $D^2 + e^i$. This, together with the optimality of X^2, we have $k_b + \sum_{t=b+1}^{j} h_t \leq k_j$. Moreover, by the choice of l, $X^1 + e^b$ is feasible for $D^1 + e^i$ and

$$\begin{array}{ll} f(D^2+e^i)-f(D^2) & = R_i(D_i^2+1)-R_i(D_i^2)-k_j+\sum_{t=i+1}^{j} h_t \\ & \le R_i(D_i^1+1)-R_i(D_i^1)-k_b+\sum_{t=i+1}^{j} h_t-\sum_{t=b+1}^{j} h_t \\ & = \begin{cases} R_i(D_i^1+1)-R_i(D_i^1)-k_b+\sum_{t=i+1}^{b} h_t & \text{if } i<b \\ R_i(D_i^1+1)-R_i(D_i^1)-k_b-\sum_{t=b+1}^{i} h_t & \text{if } i\ge b \end{cases} \\ & \le f(D^1+e^i)-f(D^1). \end{array}$$

∎

The above lemma allows us to prove the following.

Lemma 2.6 *$f(\cdot)$ satisfies (L1).*

Proof. Our objective is to show that for $D^2 \geq D^1$, if $f(D^1) \geq f(D^1+e^i)$ then $f(D^2) \geq f(D^2+e^i)$ for all e^i. We consider two cases depending on whether D^2+e^i is in the feasible region, F. If $D^2+e^i \in F$ then by Lemma 2.5 we know that $f(D^1) \geq f(D^1+e^i)$ implies $f(D^2) \geq f(D^2+e^i)$. Consider now the case when $D^2+e^i \notin F$. Since $f(D) \geq 0$ for any $D \in F$, $f(D^2+e^i) = -1 \leq f(D^2)$. Hence we are done. ∎

We can now complete the proof of Theorem 2.2 by showing that $f(\cdot)$ satisfies (L2) with respect to F.

Proof. Our objective is to show that if $D^2, D^1+e^i \in F$, $D^2 > D^1$, $D_i^2 = D_i^1$ with $f(D^1+e^1) \geq f(D^1+e^j)$ for all $D^1+e^j \in F$, then there exists $D^2+e^i-e^l \in F$ such that $f(D^2+e^i-e^l) \geq f(D^2)$ and $D_l^2 > D_l^1$.

Consider $D^1 < D^2$ with $D_i^1 = D_i^2$, $D^1+e^i \in F$, $f(D^1+e^i) \geq f(D^1+e^j)$ for all $D^1+e^j \in F$, and let $D^4 = D^1+e^i$ and $X_\beta^4 = X_\beta^1+1$ with $X^4 \geq X^1$. That is, the increase in demand at period i of D^1 as we move to D^4 is satisfied by increasing the production at period β. Recall that β may be $\leq i$ or $> i$.

We divide the proof into cases depending on the position of the production period β with respect to the demand consumption period i. For each case, we use different choices of l, which is the period in which to decrease demand by one unit as we move from D^2+e^1 to $D^2+e^i-e^l$. In fact. for each case, we must show that our choice of l is feasible and that $f(D^2+e^i-e^l) \geq f(D^2)$.

- Case I: $\beta \leq i$. In this case, we will further distinguish between two sub-cases depending on whether or not period β is producing to capacity to satisfy demand D^2.

  - Case Ia: $X_\beta^2 < Q_\beta^2$, then X^2+e^β is feasible for D^2+e^i. Since $D^1 < D^2$, then there exists an l such that $D_l^2 > D_l^1$. Let $D^5 = D^2 - e^i$ and X^5 be the corresponding production schedule. By Property 2.3, $D^5 \leq D^2$ implies we can select $X^5 \leq X^2$ and hence X^5+e^β is feasible for $D^2+e^i-e^l$.

Thus, by Lemma 2.5

$$\begin{aligned} f(D^2 + e^i - e^l) - f(D^2 - e^l) &\geq f(D^2 + e^i) - f(D^2) \\ &\geq R_i(D_i^2 + 1) - R_i(D_i^2) - k_\beta - \textstyle\sum_{t=\beta+1}^{i} h_t \\ &= f(D^1 + e^i) - f(D^1) \\ &\geq f(D^1 + e^l) - f(D^1) \\ &\geq f(D^2) - f(D^2 - e^l), \end{aligned}$$

where the last inequality is again justified by Lemma 2.5.

- Case Ib: $X_\beta^2 = Q_\beta^2$. We need to find a period l where $D_l^2 > D_l^1$. The choice of l depends on the inventory position at period β.
  - CaseIb(i): $I_\beta^2 \geq I_\beta^1$. The definition of β, together with $X_\beta^2 = Q_\beta^2$, implies that $X_\beta^2 > X_\beta^1$. Thus, using Property 2.3 we have

$$\sum_{t=\beta}^{T} D_t^2 = I_\beta^2 + \sum_{t=\beta}^{T} X_t^2 > I_\beta^1 + \sum_{t=\beta}^{T} X_t^1 = \sum_{t=\beta}^{T} D_t^1.$$

Hence, there exists l, $l \geq \beta$, with $D_l^2 > D_l^1$. In case there is more than one choice for l, we select the smallest index $\geq \beta$.
Again, since $X_\beta^2 > X_\beta^1$, then for $\alpha = \beta, \beta+1, \ldots, l-1$,

$$\begin{aligned} I_{\alpha+1}^2 &= I_\beta^2 + \sum_{t=\beta}^{\alpha} X_t^2 - \sum_{t=\beta}^{\alpha} D_t^2 \\ &> I_\beta^2 + \sum_{t=\beta}^{\alpha} X_t^1 - \sum_{t=\beta}^{\alpha} D_t^1 \\ &\geq I_\beta^1 + \sum_{t=\beta}^{\alpha} X_t^1 - \sum_{t=\beta}^{\alpha} D_t^1 \\ &= I_{\alpha+1}^1 \geq 0. \end{aligned}$$

Hence X^2 is feasible for $D^2 + e^i - e^l$. Finally, in this case we have,

$$\begin{aligned} 0 &\le f(D^1+e^i) - f(D^1+e^l) \\ &= f(D^1+e^i) - f(D^1) - [f(D^1+e^l) - f(D^1)] \\ &= R_i(D_i^2+1) - R_i(D_i^2) - k_\beta - \textstyle\sum_{t=\beta+1}^{i} h_t - [f(D^1+e^l) - f(D^1)] \\ &\le R_i(D_i^2+1) - R_i(D_i^2) - k_\beta - \textstyle\sum_{t=\beta+1}^{i} h_t \\ &\quad -[R_l(D_l^1+1) - R_l(D_l^1) - k_\beta - \textstyle\sum_{t=\beta+1}^{l} h_t] \\ &\quad \text{(since } X^1+e^\beta \text{ is feasible for } D^1+e^l\text{)} \\ &\le R_i(D_i^2+1) - R_i(D_i^2) - k_\beta - \textstyle\sum_{t=\beta+1}^{i} h_t \\ &\quad -[R_l(D_l^2) - R_l(D_l^2-1) - k_\beta - \textstyle\sum_{t=\beta+1}^{l} h_t] \\ &\quad \text{(because of the concavity of } R_l\text{)} \\ &\le f(D^2+e^i-e^l) - f(D^2) \text{ (since } X^2 \text{ is feasible for } D^2+e^i-e^l\text{).} \end{aligned}$$

∗ Case Ib(ii): $I_\beta^2 < I_\beta^1$, which implies that

$$\sum_{t=1}^{\beta-1} D_t^2 = -I_\beta^2 + \sum_{t=1}^{\beta-1} X_t^2 > -I_\beta^1 + \sum_{t=1}^{\beta-1} X_t^1 = \sum_{t=1}^{\beta-1} D_t^1.$$

Hence, there exists $l, < \beta$, with $D_l^2 > D_l^1$. In case there is more than one choice for l we select the largest one with index $< \beta$. Thus, for $\alpha = l, l+1, \ldots, \beta-1$, we have $D_\alpha^1 = D_\alpha^2$ and therefore,

$$I_{\alpha+1}^1 = I_\beta^1 + \sum_{t=\alpha+1}^{\beta-1} D_t^1 - \sum_{t=\alpha+1}^{\beta-1} X_t^1 > I_\beta^2 + \sum_{t=\alpha+1}^{\beta-1} D_t^2 - \sum_{t=\alpha+1}^{\beta-1} X_t^2 = I_{\alpha+1}^2 \ge 0.$$

This implies that $X^1 + e^\beta$ is feasible for $D^1 + e^l$, and since $l < \beta \le i$, X^2 is feasible for $D^2 + e^i - e^l$. Finally, in this case we have,

$$\begin{aligned} 0 &\le f(D^1+e^i) - f(D^1+e^l) \\ &= f(D^1+e^i) - f(D^1) - [f(D^1+e^l) - f(D^1)] \\ &= R_i(D_i^2+1) - R_i(D_i^2) - k_\beta - \textstyle\sum_{t=\beta+1}^{i} h_t - [f(D^1+e^l) - f(D^1)] \\ &\le R_i(D_i^2+1) - R_i(D_i^2) - k_\beta - \textstyle\sum_{t=\beta+1}^{i} h_t \\ &\quad -[R_l(D_l^1+1) - R_l(D_l^1) - k_\beta + \textstyle\sum_{t=l+1}^{\beta} h_t] \\ &\quad \text{(since } X^1+e^\beta \text{ is feasible for } D^1+e^l\text{)} \\ &\le R_i(D_i^2+1) - R_i(D_i^2) - k_\beta - \textstyle\sum_{t=\beta+1}^{i} h_t \\ &\quad -[R_l(D_l^2) - R_l(D_l^2-1) - k_\beta + \textstyle\sum_{t=l+1}^{\beta} h_t] \\ &\quad \text{(because of the concavity of } R_l\text{)} \\ &\le f(D^2+e^i-e^l) - f(D^2) \text{ (since } X^2 \text{ is feasible for } D^2+e^i-e^l\text{).} \end{aligned}$$

- Case II: $\beta > i$, then $I_{t+1}^1 > 0$ for $t = i, i+1, \ldots, \beta-1$. In this case, we need to consider the inventory positions between periods i and β in order to choose period l.

  – Case IIa: $I_{t+1}^2 \ge I_{t+1}^1$ for $t = i, i+1, \ldots, \beta-1$. Again, we divide the proof into cases based on whether or not period β is producing to capacity to satisfy demand D^2.

* Case IIa(i): $X^2_\beta < Q^2_\beta$, then $X^2 + e^\beta$ is feasible for $D^2 + e^i$. Since $D^1 < D^2$, there exists an l such that $D^2_l > D^1_l$. Moreover,

$$\begin{aligned} f(D^2) - f(D^2 - e^l) &\leq f(D^1 + e^l) - f(D^1) \qquad \text{(by Lemma 2.5)} \\ &\leq f(D^1 + e^i) - f(D^1) \\ &= R_i(D^1_i + 1) - R_i(D^1_i) - k_\beta + \textstyle\sum_{t=i+1}^{\beta} h_t \\ &\leq f(D^2 + e^i) - f(D^2) \\ &\quad \text{(since } X^2 + e^\beta \text{ is feasible for } D^2 + e^i\text{)} \\ &\leq f(D^2 + e^i - e^l) - f(D^2 - e^l) \\ &\quad \text{(by Lemma 2.5).} \end{aligned}$$

* Case IIa(ii): $X^2_\beta = Q^2_\beta$. Since $X^2_\beta > X^1_\beta$, then

$$\sum_{t=\beta}^{T} D^2_t = I^2_\beta + \sum_{t=\beta}^{T} X^2_t > I^1_\beta + \sum_{t=\beta}^{T} X^1_t = \sum_{t=\beta}^{T} D^1_t.$$

Hence, there exists l, $l \geq \beta$, with $D^2_l > D^1_l$. If there is more than one choice for l, we select the smallest one. Since $X^2_\beta > X^1_\beta$, we have for $\alpha = \beta, \beta{+}1, \ldots, l{-}1$,

$$\begin{aligned} I^2_{\alpha+1} &= I^2_\beta + \sum_{t=\beta}^{\alpha} X^2_t - \sum_{t=\beta}^{\alpha} D^2_t \\ &> I^1_\beta + \sum_{t=\beta}^{\alpha} X^1_t - \sum_{t=\beta}^{\alpha} D^1_t \\ &= I^1_{\alpha+1} \geq 0. \end{aligned}$$

Hence $X^2 - e^\beta$ is feasible for $D^2 - e^l$ and X^2 is feasible for $D^2 + e^i - e^l$. Thus, in this case we have,

$$\begin{aligned} &f(D^2 + e^i - e^l) - f(D^2) \\ &\geq R_i(D^2_i + 1) - R_i(D^2_i) - k_\beta + \textstyle\sum_{t=i+1}^{\beta} h_t \\ &\qquad -[R_l(D^2_l) - R_l(D^2_l - 1) - k_\beta - \textstyle\sum_{t=\beta+1}^{l} h_t] \\ &\quad \text{(since } X^2 \text{ is feasible for } D^2 + e^i - e^l\text{)} \\ &= f(D^1 + e^i) - f(D^1) - [R_l(D^2_l) - R_l(D^2_l - 1) - k_\beta - \textstyle\sum_{t=\beta+1}^{l} h_t] \\ &\geq f(D^1 + e^i) - f(D^1) - [R_l(D^1_l + 1) - R_l(D^1_l) - k_\beta - \textstyle\sum_{t=\beta+1}^{l} h_t] \\ &\quad \text{(because of the concavity of } R_l\text{)} \\ &\geq f(D^1 + e^i) - f(D^1) - [f(D^1 + e^l) - f(D^1)] \\ &\quad \text{(since } l \geq \beta \text{ implies that } X^1 + e^\beta \text{ is feasible for } D^1 + e^l\text{)} \\ &\geq 0. \end{aligned}$$

– Case IIb: $I^2_{t+1} < I^1_{t+1}$, for some $t = i, i+1, \ldots, \beta-1$. Let γ be the smallest index $\geq i$ with $I^2_{\gamma+1} < I^1_{\gamma+1}$. Then,

$$\sum_{t=1}^{\gamma} D^2_t = \sum_{t=1}^{\gamma} X^2_t - I^2_{\gamma+1} > \sum_{t=1}^{\gamma} X^1_t - I^1_{\gamma+1} = \sum_{t=1}^{\gamma} D^1_t.$$

Let l be the largest index $\leq \gamma$ with $D^2_l > D^1_l$. Since $I^2_{\gamma+1} < I^1_{\gamma+1}$, we have for $\alpha = l, l+1, \ldots, \gamma-1$,

$$\begin{aligned} I^1_{\alpha+1} &= I^1_{\gamma+1} - \sum_{t=\alpha+1}^{\gamma} X^1_t + \sum_{t=\alpha+1}^{\gamma} D^1_t \\ &> I^2_{\gamma+1} - \sum_{t=\alpha+1}^{\gamma} X^2_t + \sum_{t=\alpha+1}^{\gamma} D^2_t \\ &= I^2_{\alpha+1} \geq 0. \end{aligned}$$

Hence $X^1 + e^\beta$ is feasible for $D^1 + e^l$. Finally,

$$\begin{aligned} 0 \quad &\leq f(D^1+e^i) - f(D^1) - [f(D^1+e^l) - f(D^1)] \\ &\leq R_i(D^1_i+1) - R_i(D^1_i) - k_\beta + \textstyle\sum_{t=i+1}^{\beta} h_t \\ &\quad -[R_l(D^1_l+1) - R_l(D^1_l) - k_\beta + \textstyle\sum_{t=l+1}^{\beta} h_t] \\ &\quad (\text{since } X^1+e^\beta \text{ is feasible for } X^1+e^l) \\ &\leq R_i(D^2_i+1) - R_i(D^2_i) - k_\beta + \textstyle\sum_{t=i+1}^{\beta} h_t \\ &\quad -[R_l(D^2_l) - R_l(D^2_l-1) - k_\beta + \textstyle\sum_{t=l+1}^{\beta} h_t] \\ &\quad (\text{because } D^2_l > D^1_l) \\ &\leq f(D^2+e^i-e^l) - f(D^2) \end{aligned}$$

since $l \leq i$ implies that X^2 is feasible for $D^2 + e^i - e^l$.

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2.5 Model Extensions and Limitations

There are a number of additional applications that the model covers. Below we list some that have lightly concave cost functions over a polymatroid feasible region, and thus can be solved with the greedy algorithm.

1. *Multi-product systems* The model presented in this paper may be extended to cover supply chains with m, $m \geq 1$, products each of which is assembled from a set of parts and shares common production capacity. We assume that revenue curves exist for each product at each time period and that there are no demand diversions among different products.

2. *Variable Leadtimes:* Consider models in which a customer places an order in period j, $j = 1, 2, \ldots, T$, and the product is delivered within L_j time periods. In this case, D_t represents demand satisfied at time t, $t = 1, 2, \ldots, T$. In Problem PP, the revenue function of demand would need to be modified to account for all periods in which an order could have been placed to be satisfied by a delivery in time period t.

3. *Different Classes of Customers:* In practice, customers are sometimes distinguished by their responsiveness to different leadtimes. For instance, consider a model with two types of customers; customers who insist on receiving the product immediately and those who are willing to wait up to L time periods. This problem fits in the class of problems described in this paper if the firm has information on the demand curves by customer type and wait time, and the revenue functions are concave.

4. *Production Set-up Cost:* An important extension of the model is the addition of production set-up cost. In this case, it is possible to show that an optimal policy is *consecutive*. That is, in an optimal policy, production in a specific period, say t, will be used to satisfy demand in consecutive periods, say periods $r, r+1, r+2, \ldots, s$. However, due to production capacity limits, it is not true that production in period t will necessarily satisfy all of the demand in periods r and s. Indeed, this observation leads to a dynamic program where the states are the initial inventory levels in the time periods, and the greedy algorithm is used within the dynamic program to determine the optimal demands satisfied between periods r and s given the initial inventory levels of those two periods. Of course, this algorithm is exponential and may not be efficient for large problem instances.

There are a number of limitations to the model that we consider. For example, in this paper we consider dynamic pricing in a monopolistic scenario. Clearly a more realistic scenario accounts for price competition. To date, few models have examined dynamic pricing under systemic constraints such as production capacity while also considering competition; this represents an ongoing area of challenge and interest. In addition, our research in this paper does not consider strategic buying practices, where customers time their purchases according to the price. Indeed, reality suggests that some customers will purchase strategically while others purchase according to immediate need. Incorporating this behavior in a multi-period setting is also a significant challenge but one of interest to researchers and industry alike. Finally, the model we present assumes that customer demand behaves according to deterministic demand curves. Adding a stochastic component to demand is an ongoing research interest of ours; more discussion of our stochastic pricing models is available in Swann (2001).

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