Proofs for "Policies Utilizing Tactical Inventory for Service-Differentiated Customers"

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Abstract

We consider a manufacturer serving two customer classes where one wants the item immediately and the second receives a discount to accept a delay. We show that a (S, R, B) base stock policy is optimal under differentiation and non-differentiation where S, R, and B are the order-up-to, reserve-up-to, and backlog-up-to amounts.

The body of the paper can be found in Duran et al. [3]. The lemmas are summarized below with corresponding proofs.

1 Proof of Lemmas

Lemma 1. *In any optimal policy under the Time Differentiation Strategy, we have:*

$$B_t \cdot (R_t^1 + R_t^2) = 0 \qquad t = 1, 2, \ldots, T.$$

Proof. The simplification is similar in structure to the proof for simplification of CSS below but longer so the full details are not listed. The main difference is that TDS includes reserving decisions for each customer class so there are more cases to consider. It is also similar to a simplification proof of Lemma 1 in Duran et al. [2] but with fewer cases since both classes do not accept delayed service. ☐

Lemma 2. *In any optimal policy under the Common Service Strategy, we have* $R_t \cdot B_t = 0$*, for* $t = 1, 2, ..., T$.

Proof. By contradiction, assume that there is an optimal policy with $R_t \cdot B_t > 0$ for some period t. Let $\overline{R_t} = R_t - 1$ and $\overline{B_t} = B_t - 1$ be the alternative policy, and let V_t and $\overline{V_t}$ be the expected profit starting from period t under the two policies respectively. We compare the two policies in the following three cases:

- Case 1: $D_t^{1,2} \leq S_t - R_t$, hence $D_t^{1,2} < S_t - \overline{R_t}$.

$$V_t = p_t^2 D_t^{1,2} - h_t(S_t - D_t^{1,2}) + J_{t+1}^{CSS}(S_t - D_t^{1,2}) = \overline{V_t}$$

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- Case 2: $S_t - R_t + B_t/\alpha_t > D_t^{1,2} > S_t - R_t$, hence $S_t - \overline{R_t} + \overline{B_t}/\alpha_t \geq D_t^{1,2} \geq S_t - \overline{R_t}$.

  $$\begin{aligned} V_t &= p_t^2(S_t - R_t + \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor) - h_t R_t - \beta_t^2 \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor \\ &\quad -\ell_t^1 \lceil (1-\alpha_t)(D_t^{1,2} - S_t + R_t)\rceil + J_{t+1}^{CSS}(R_t - \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor) \\ \overline{V_t} &= p_t^2(S_t - R_t + 1 + \lfloor \alpha_t(D_t^{1,2} - S_t + R_t - 1)\rfloor) - h_t(R_t - 1) - \beta_t^2 \lfloor \alpha_t(D_t^{1,2} - S_t + R_t - 1)\rfloor \\ &\quad -\ell_t^1 \lceil (1-\alpha_t)(D_t^{1,2} - S_t + R_t - 1)\rceil + J_{t+1}^{CSS}(R_t - 1 - \lfloor \alpha_t(D_t^{1,2} - S_t + R_t - 1)\rfloor) \end{aligned}$$

  If $\lfloor \alpha_t(D_t^{1,2} - S_t + R_t - 1)\rfloor = \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor$, then $\lceil (1-\alpha_t)(D_t^{1,2} - S_t + R_t - 1)\rceil = \lceil (1-\alpha_t)(D_t^{1,2} - S_t + R_t)\rceil - 1$. We have,

  $$\overline{V_t} = V_t + p_t^2 + h_t + \ell_t^1 - J_{t+1}^{CSS}(R_t - \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor) + J_{t+1}^{CSS}(R_t - 1 - \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor)$$

  Since $D_t^{1,2} < S_t - R_t + B_t/\alpha_t$, a new demand from class 2 will be accepted, which means $p_t^2 + h_t + \ell_t^2 + J_{t+1}^{CSS}(R_t - 1 - \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor) \geq J_{t+1}^{CSS}(R_t - \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor)$. Thus $\overline{V_t} \geq V_t$.

  Otherwise, $\lfloor \alpha_t(D_t^{1,2} - S_t + R_t - 1)\rfloor = \lfloor \alpha_t(D_t^{1,2} - S_t + R_t)\rfloor$-1, then $\lceil (1-\alpha_t)(D_t^{1,2} - S_t + R_t - 1)\rceil = \lceil (1-\alpha_t)(D_t^{1,2} - S_t + R_t)\rceil$. We have, $\overline{V_t} = V_t + h_t + \beta_t^2 \geq V_t$.

- Case 3: $D_t^{1,2} \geq S_t - R_t + B_t/\alpha_t$, hence $D_t^{1,2} > S_t - \overline{R_t} + \overline{B_t}/\alpha_t$.

  $$\begin{aligned} V_t &= p_t^2(S_t - R_t + B_t) - h_t R_t - \beta_t^2 B_t - \ell_t^1(1-\alpha_t)B_t/\alpha_t - \ell_t(D_t^{1,2} - S_t + R_t - B_t/\alpha_t) \\ &\quad + J_{t+1}^{CSS}(R_t - B_t) \\ \overline{V_t} &= V_t + h_t + \beta_t^2 + (1-\alpha_t)(\ell_t^1 - \ell_t^2) \geq V_t \end{aligned}$$

The expected profit under the alternative policy is always greater or equal to that under the current policy, which incurs a contradiction. □

Lemma 3. *Given $g(x,y)$ is jointly concave in x and y, $G(x) = \max_y g(x,y)$ is a concave function for x.*

This lemma will be used in the proofs of the theorems below. Please see Duran et al. [2] for the proof.

2 Proof of Concavity Results

Theorem 4. *For all $t = 1, ..., T$,*

- *$g_t^{TDS}(S_t, R_t^1, R_t^2, 0)$ is a jointly quasi-concave function of R_t^1 and R_t^2, and $g_t^{CSS}(S_t, R_t, 0)$ is a quasi-concave function of R_t,*
- *$g_t^{TDS}(S_t, 0, 0, B_t)$ and $g_t^{CSS}(S_t, 0, B_t)$ are quasi-concave functions of B_t,*
- *$G_t^{TDS}(S_t)$ and $G_t^{CSS}(S_t)$ are concave functions of S_t,*
- *$J_t^{TDS}(I_t)$ and $J_t^{CSS}(I_t)$ are concave functions of I_t,*
- *The unconstrained optimizers ($R_t^{1^*}$, $R_t^{2^*}$, and B_t^*) for functions $g_t^{TDS}(S_t, R_t^1, R_t^2, 0)$, and $g_t^{TDS}(S_t, 0, 0, B_t)$, are independent of inventory level S_t, where for $i = 1, 2$,*

  $$(R_t^{1^*}(S_t), R_t^{2^*}(S_t)) = \underset{(R_t^1, R_t^2): 0 \leq R_t^1, 0 \leq R_t^2}{\operatorname{argmax}} \left\{ g_t^{TDS}(S_t, R_t^1, R_t^2, 0) \right\} \text{ and } B_t^*(S_t) = \underset{B_t: 0 \leq B_t}{\operatorname{argmax}} \left\{ g_t^{TDS}(S_t, 0, 0, B_t) \right\}.$$

- *The unconstrained optimizers (R_t^* and B_t^*) for $g_t^{CSS}(S_t, R_t, 0)$ and $g_t^{CSS}(S_t, 0, B_t)$ are independent of inventory level S_t, where*

$$R_t^*(S_t) = \underset{R_t : 0 \le R_t}{\operatorname{argmax}} \Big\{ g_t^{CSS}(S_t, R_t, 0) \Big\}, B_t^*(S_t) = \underset{B_t : 0 \le B_t}{\operatorname{argmax}} \Big\{ g_t^{CSS}(S_t, 0, B_t) \Big\}.$$

The structural results imply the form of the optimal decisions. For example, the optimal decisions for CSS are defined by the following:

$$\begin{aligned} S_t^* &= \max\{S : c_t \le G_t^{\prime CSS}(S)\} & & if\; c_t \le G_t^{\prime CSS}(0) \\ R_t^* &= \max\{I : p_t^2 + \ell_t + h_t \le J_{t+1}^{\prime CSS}(I)\} & & if\; p_t^2 + \ell_t + h_t < J_{t+1}^{\prime CSS}(0) \\ B_t^* &= \min\{I : J_{t+1}^{\prime CSS}(-I) \ge p_t^2 + \ell_t^2 - \beta_t^2\} & & if\; p_t^2 + \ell_t^2 - \beta_t^2 > J_{t+1}^{\prime CSS}(0), \end{aligned} \tag{1}$$

and the optimal decisions for TDS that are different from CSS are given by:

$$\begin{aligned} R_t^{1^*} &= \max\{I : p_t^1 + \ell_t^1 + h_t \le J_{t+1}^{\prime TDS}(I)\} & & if\; p_t^1 + \ell_t^1 + h_t < J_{t+1}^{\prime TDS}(0) \\ R_t^{1^*} + R_t^{2^*} &= \max\{I : p_t^2 + \ell_t^2 + h_t \le J_{t+1}^{\prime TDS}(I)\} & & if\; p_t^2 + \ell_t^2 + h_t < J_{t+1}^{\prime TDS}(0). \end{aligned} \tag{2}$$

2.1 For the Time Differentiation Strategy

Proof. Proof is very similar in structure to the case for Common Service Strategy (which is shorter and thus shown in its entirety below). It follows the same seven steps using the profit-to-go functions of the Time Differentiation Strategy, so the derivatives are somewhat different (see [1] for details). ☐

2.2 For the Common Service Strategy

In the proof below, the CSS superscript is omitted from the expected profit functions to increase readability.

Proof. Let $j_t(I_t, S_t) = -c_t(S_t - I_t) + G_t(S_t)$, so $J_t(I_t) = \max\limits_{S_t : I_t \le S_t \le I_t + q_t} j_t(I_t, S_t)$. We prove by induction.

1. For period $t = T$, we have $B_T = 0$, $R_T = 0$ and $J_{T+1}(I_T) = v \cdot I_T$.
   $G_T(S_T)$ is concave in S_T, since $G_T'(S_T)$ is non-increasing in S_T:
   $G_T''(S_T) = (v - h_T - p_T^2 - \ell_T)\phi_T^{1,2}(S_t) \le 0$, since $v < p_T^2$.

2. Given $t + 1 \le T$, assume that $G_{t+1}(S_{t+1})$ is concave in S_{t+1}, then it is easy to see that $j_{t+1}(I_{t+1}, S_{t+1})$ is jointly concave in I_{t+1} and S_{t+1}. So by Lemma 3, $J_{t+1}(I_{t+1})$ is concave in I_{t+1}, and as a result $J_{t+1}'(I_{t+1})$ is non-increasing in I_{t+1}.

3. Next let us prove that $g_t^{CSS}(S_t, R_t, 0)$ is quasi-concave in R_t. We have,

   $$\frac{\partial g_t^{CSS}(S_t, R_t, 0)}{\partial R_t} = (-p_t^2 - \ell_t - h_t + J_{t+1}^{\prime}(R_t))(1 - \Phi_t^{1,2}(S_t - R_t)) \qquad if\; S_t \ge R_t (= 0\; otherwise).$$

   If R_t^* is defined as in (1), we have $g_t^{\prime CSS}(S_t, R_t, 0) \ge 0$ when $0 \le R_t \le R_t^*$, and $g_t^{\prime CSS}(S_t, R_t, 0) \le 0$ when $R_t > R_t^*$; thus, $g_t^{CSS}(S_t, R_t, 0)$ is quasi-concave with respect to R_t. R_t^* is the unique unconstrained optimizer of $g_t^{CSS}(S_t, R_t, 0)$, and it is independent of inventory level S_t. $R_t^c = \min(R_t^*, S_t)$ maximizes $g_t^{CSS}(S_t, R_t, 0)$, for $0 \le R_t \le (S_t)^+$.

4. Next let us prove that $g_t^{CSS}(S_t, 0, B_t)$ is quasi-concave in B_t. Taking the derivative,

   $$\frac{\partial g_t^{CSS}(S_t, 0, B_t)}{\partial B_t} = \int_{S_t+B_t/\alpha_t}^{\infty} [p_t^2 - \beta_t^2 + \ell_t^2 - J'_{t+1}(-B_t)] d\mathbf{\Phi}_t^{1,2}(D_t^{1,2}).$$

   Let us define B_t^* as in (1), then we have $g_t^{'CSS}(S_t, 0, B_t) \geq 0$ when $0 \leq B_t \leq B_t^*$, and $g_t^{'CSS}(S_t, 0, B_t) \leq 0$ when $B_t > B_t^*$; thus, $g_t^{CSS}(S_t, 0, B_t)$ is quasi-concave with respect to B_t. B_t^* is the unique unconstrained optimizer of $g_t^{CSS}(S_t, 0, B_t)$, and it is independent of inventory level S_t. $B_t^c = \min(B_t^*, q_{t+1})$ maximizes $g_t^{CSS}(S_t, 0, B_t)$, for $0 \leq B_t \leq q_{t+1}$.

5. Let us prove the concavity of $G_t^R(S_t)$ with respect to S_t.

   We consider $G_t^{''R}(S_t)$ in three cases:
   (a)Case 1: $S_t < R_t^*$:
   $G_t^{''R}(S_t) = J''_{t+1}(S_t) \leq 0$ due to the concavity of J_{t+1}.
   (b)Case 2: $S_t > R_t^*$:
   $G_t^{''R}(S_t) = \int_0^{S_t - R_t^*} J''_{t+1}(S_t - k) d\mathbf{\Phi}_t^{1,2}(k) + (J'_{t+1}(R_t^*) - p_t^2 - \ell_t - h_t)\phi_t^{1,2}(S_t - R_t^*) \leq 0$
   due to the choice of R_t^* and the concavity of J_{t+1}.
   (c)Case 3: $S_t = R_t^*$:
   $G_t^{'R}(R_t^*+) - G_t^{'R}(R_t^*-) = p_t^2 + \ell_t + h_t - J'_{t+1}(R_t^*) \leq 0$
   due to the choice of R_t^* and the concavity of J_{t+1}.

   Since $G_t^{''R}(S_t) \leq 0$ for all S_t, $G_t^R(S_t)$ is concave in S_t.

6. Let us prove the concavity of $G_t^B(S_t)$ with respect to S_t, where $G_t^B(S_t) = g_t^{CSS}(S_t, 0, B_t^c)$. We have,

   $$\begin{aligned} G_t^{''B}(S_t) \quad = \quad & -\alpha_t(p_t^2 + \ell_t^2 - \beta_t^2 - J'_{t+1}(-B_t))\phi_t^{1,2}(S_t + B_t/\alpha_t) \\ & +[\alpha_t(p_t^2 + \ell_t^1 - \beta_t^2 - J'_{t+1}(0)) - (p_t^2 + \ell_t^1 + h_t - J'_{t+1}(0))]\phi_t^{1,2}(S_t) \\ & + \textstyle\int_0^{S_t} J''_{t+1}(S_t - k) d\mathbf{\Phi}_t^{1,2}(k) + \int_{S_t}^{S_t + B_t/\alpha_t} \alpha_t^2 J''_{t+1}(\alpha_t(S_t - k)) d\mathbf{\Phi}_t^{1,2}(k). \end{aligned}$$

   The first term in $G_t^{''B}(S_t)$ is negative due to the choice of B_t^*. The third and the fourth terms in $G_t^{''B}(S_t)$ are negative due to the concavity of $J_{t+1}(S_t)$. We have $G_t^{''B}(S_t) \leq 0$ and therefore, $G_t^B(S_t, B_t)$ is concave in S_t.

7. Let us prove the concavity of $G_t(S_t)$.
   In each period, we must be in one of the following cases, which are independent of the S_t values:

   - If $p_t^2 + \ell_t^2 - \beta_t^2 \leq J'_{t+1}(0) \leq p_t^2 + \ell_t + h_t$, we have $R_t^* = B_t^* = 0$, therefore $R_t^c = B_t^c = 0$; thus, we have $G_t(S_t) = G_t^R(S_t) = G_t^B(S_t)$.
   - If $J'_{t+1}(0) > p_t^2 + \ell_t + h_t$, we have $R_t^* \geq 0$ and $B_t^* = 0$, therefore $R_t^c \geq 0$ and $B_t^c = 0$; thus, we have $G_t(S_t) = G_t^R(S_t) \geq G_t^B(S_t)$.
   - If $p_t^2 + \ell_t^2 - \beta_t^2 < J'_{t+1}(0)$, we have $B_t^* \geq 0$ and $R_t^* = 0$, therefore $B_t^c \geq 0$ and $R_t^c = 0$; thus, we have $G_t(S_t) = G_t^B(S_t) \geq G_t^R(S_t)$.

   We see that in each period, $G_t(S_t)$ reduces to some function that is proved to be concave. Therefore $G_t(S_t)$ is concave.

□

Corollary 5. *Given a vector of prices, there exists an optimal policy for*

- *the Time Differentiation Strategy with an optimal order-up-to level (S_t^*), optimal reserve-up-to-levels ($R_t^{1^*}$ and $R_t^{2^*}$), and an optimal backlog-up-to level (B_t^*),*
- *the Common Service Strategy with an optimal order-up-to level (S_t^*), an optimal reserve-up-to-level (R_t^*) and an optimal backlog-up-to level (B_t^*).*

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References

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