Dynamic Switching Times for Season and Single Tickets in Sports and Entertainment*

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Abstract

Revenue management can be used in many industries where there is a limited, perishable capacity and the market can be segmented. In this paper we focus on the sales of event tickets in the Sports and Entertainment industries, where tickets are sold exclusively as season tickets initially or as single events later in the selling horizon. We specifically study the optimal time to switch between these market segments dynamically as a function of the state of the system. Under Poisson demand processes, we find the optimal switching time is a set of time thresholds that depends on the remaining inventory and time left in the horizon. We generalize the model and results to apply to sales of ℓ ($\ell > 2$) events in the season or to incorporate demand rates that depend on time. We use numerical experiments to show that the profit improvement can be 1 - 2% over optimal static switching and that dynamic switching can reduce profit variability.

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1 Introduction

Revenue Management (RM) is a set of tools to help mathematically determine decisions such as the right prices or inventory to make available so as to maximize profit. Revenue Management has made great strides in improving the bottom line of many firms, especially in airlines (Smith et al. [22]), hotels (Lieberman [16]), and rental car agencies (Geraghty and Johnson [12]), where RM is recognized as a key factor in the firm's viability and success.

However many other industries offer a rich set of RM-type problems that have not been fully addressed. One of these is the sports and entertainment (S&E) industry, where tickets are sold to events at a venue such as a sports stadium or theater. Like the airlines, the capacity for an event is generally fixed, there are high fixed operational costs and low marginal costs for additional sales, and the market can be segmented by customer type. In S&E, one important market segmentation

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is that some customers buy season packages, or bundles of tickets to events during the season, while others buy individual tickets to events. Season ticket holders are important to the success of the organization, since they are more likely to donate to the organization or renew tickets in the future. They are also desirable since they commit to a bundle of tickets in advance, which can offer greater cash flow to an organization or commitments upon which to base future decisions. Most S&E firms offer season tickets first, and they open purchasing for single tickets at a later date but before the start of the season. A basic trade-off is that the firms want to capture significant demand for bundled tickets while still allowing enough time for individual ticket purchases when bundle demand alone will not sell out the event, as it often does not.

In this paper, we study the specific question of timing the switch from selling bundles to selling single tickets. This is a problem motivated by our discussions with several large S&E firms, where the timing of the sales, or in some cases, the timing of the promotion of sales to the public is of key interest. The problem we analyze also is relevant to other industries where revenue management applies. For instance, many hotels make accommodations for group bookings for weddings or conferences, but the commitments must be made in advance and the unsold rooms are released to the general public in advance of the travel date for bookings by individuals. It is also possible to sell bundled capacity and smaller units of capacity in industries such as manufacturing, where contracts may be negotiated with prioritized clients for large volumes of capacity sold in advance.

A key aspect to the problem we study is that the same limited capacity can be sold as either bundled and single tickets. In addition, after sales of single tickets are allowed, there may be multiple events for sale simultaneously. These characteristics, along with the desire to *dynamically* determine the timing decision when demand is stochastic, implied that new research was needed to solve this problem. We find that the structure of the problem leads to an optimal timing policy that is relatively easy to understand and implement. The resulting policy is defined by a set of threshold pairs of times and remaining inventory, which determine the switch from bundles to single ticket sales. After each bundle sale, if the current time is less than the corresponding threshold, then the switch is made to selling individual tickets. We describe an algorithm that will compute the threshold pairs, and we demonstrate the value of the dynamic timing decision. We are able to generalize our results in several ways, including allowing the demand rates for the bundles and single-tickets to depend on time.

In the next section we describe the relevant literature to the dynamic timing problem. In Section 3, we introduce the assumptions and the model, and we present key results for the base case. We generalize the model in several ways in Section 4, and we demonstrate some numerical experiments in Section 5. Finally, we conclude in Section 6.

2 Literature Review

In the airlines, revenue management research include how to determine the overbooking levels (Littlewood [17]), seat allocations by class (Belobaba [1]), or the bid-prices for each leg of a network (Talluri and van Ryzin [23]); recent applications in airline RM include Bertsimas and Popescu [2] and Karaesmen and Ryzin [13]. Unlike S&E, the airline industry has a network structure where demand is for an origin and destination pair, which may include multiple choices of paths for the consumer. When group purchases are considered in the airlines, they are primarily for groups of individuals purchasing tickets on one plane, rather than a single individual purchasing multiple tickets over time, and there is very limited literature on group sales as stated in Farley [7]. The most similar sale to season tickets is the offering of "flexible products", where a single individual buys the option of two or more flights at a time and assigned to one of them later by the carrier (Gallego and Phillips [11]).

There have been several papers in RM of airline and retail industries that focus on pricing as a function of time. However, in the S&E industry, most organizations keep prices as announced throughout the selling period, which is known as price stickiness in the entertainment industry (see Courty [3]). Therefore pricing as used in retailing is less applicable to the entertainment industry. In S&E, timing of different kinds of products is more common than timing of a price change.

The most relevant area in the literature is that of RM timing problems, and a related work to the current study is Feng and Gallego [9], which determines the optimal dynamic time to switch from one predetermined price to a second predetermined price so as to maximize revenue by selling a given stock over a finite time horizon. Demand is assumed to be stochastic and the demand rate is higher for the lower price, and the optimal timing policy is shown to be a time threshold depending on the remaining stock amount. The restrictions of one price change and time-invariant demand intensities are relaxed in Feng and Gallego [8], and an efficient algorithm to find the optimal value functions and the optimal pricing policy is provided. Like the latter papers, we use pre-determined prices, but a main difference in our work is that we focus on switching from selling bundles of tickets to single tickets. A second important factor in comparison to [9] and follow-on papers is that when we switch to selling singles, the bundles split into multiple simultaneous processes thus the results do not follow directly from their work. Petruzzi and Monahan [19] study optimal timing of ending a primary selling season to sell in a secondary market, where the first demand process is stochastic.

We study an initial version of the switching problem between bundles and singles in S&E in Drake et al. [5]. However, in that paper, we specifically focus on a static timing decision, as is done in some organizations, where the switching date to single tickets is announced in advance to the public, and we consider demand with the specialized form of the linear Markovian death process. In the current paper, we study the *dynamic* switching time, where the time may be determined by the sales-to-date under general Poisson demand processes, thus the techniques for analysis are quite different.

It is also important to point out that there has been analysis related to improving revenue in S&E in other disciplines. A number of papers have looked at pricing decisions within venue but did not consider bundling, for example, Leslie [15] and Rosen and Rosenfield [21] studied revenue-maximizing ticket prices for different seat qualities. Work in the economics and marketing literature that considers the selling of bundled commodities includes Venkatesh and Kamakura [24], and McAfee et al. [18] among others, but these papers focus on the pricing or grouping of the bundles, not the timing of decisions.

3 The Dynamic Timing Problem

3.1 Assumptions

Let $M \in \mathbb{Z}^+$ be the number of seats available for sale for each event, and $T \in \mathbb{R}^+$ be the selling period. In the S&E data we have seen, season tickets are rarely bought after the season begins, and the switch to selling singles is also made before the season starts in every organization with whom we have worked. Thus, we focus on the selling horizon before the season begins and assume that the selling period ends when the first event takes place. The selling period begins with first offering tickets as a bundle at price p_B and then switching to selling event tickets individually at p_i for event i, for $i = 1, 2$. (Initially, we consider the problem when there are only two events in the selling period, but we will relax this assumption in Section 4.) We assume that the product prices are predetermined at the beginning of the selling season, which is true for most organizations, especially during the time preceding the start of the season.

We assume that market segments (bundles and singles) are independent. This is supported by discussions with professional sports teams (Depaoli [4]), and it is also a common assumption for many models in revenue management. We initially assume constant demand rates with time for each of the bundled and single-ticket processes; in Section 4, we extend the model to allow demand to depend on time, which can also be used to proxy substitution among segments.

We assume that for each product, there is a corresponding Poisson process of demand: $N_B(s)$, $0 \leq s \leq t$, with known constant intensity λ_B for the bundled events; $N_1(s)$, $0 \leq s \leq t$, with known constant intensity λ_1, and $N_2(s)$, $0 \leq s \leq t$, with known constant intensity λ_2 for the two single events, respectively. The state of the system is indicated by the elapsed time t and the remaining inventory level at time t, $n(t)$.

We define $r_B = \lambda_B p_B$ and $r_i = \lambda_i p_i$, $i = 1, 2$ as the revenue rate for the bundled and individual ticket sales products, respectively. We assume that the expected revenue rate for the bundle is higher than the sum of the expected revenue rates of the single tickets, i.e., $r_B > r_1 + r_2$. Otherwise, switching immediately would be optimal for all states. This assumption can also be intuitively validated by the fact that the revenue for each bundle sale can include intangibles such as donations to the organization (sometimes required for season ticket purchases) or the value of early commitment and guaranteed revenues.

3.2 Model and Results

In this section, we will examine the value of switching now or later, measure the potential gain from delaying the switch, and use this quantification to find characteristics of the optimal time to switch. We then find a function that allows us to compute the optimal switch times, and we use this function to show the structure of the optimal switching times.

The expected total revenue from bundle and single ticket sales over the time horizon $[t, T]$ with the optimal switching time is given by $V(t, n(t))$:

$$V(t, n(t)) = \sup_{\tau \in \mathcal{T}} E\big[p_B\big((N_B(\tau) - N_B(t)) \wedge n(t)\big) + \Pi(\tau, n(\tau))\big],$$

where $\mathcal{T}$ is the set of switching times τ satisfying $t \leq \tau \leq T$ and $n(\tau) = [n(t) - N_B(\tau) + N_B(t)]^+$, where $x^+ = \max\{0, x\}$. The expected total revenue also includes the expected revenue from single ticket sales after the switch from bundles with $n(\tau)$ items available for sale over $[\tau, T]$, which is denoted as $\Pi(\tau, n(\tau))$. It is given by

$$\Pi(\tau, n(\tau)) = p_1 E[(N_1(T) - N_1(\tau)) \wedge n(\tau)] + p_2 E[(N_2(T) - N_2(\tau)) \wedge n(\tau)],$$

where $(x \wedge y)$ indicates the minimum of the x and y. The sharing of capacity of bundles and singles is reflected in the definition of $n(\tau)$.

At time t, if we can compare the expected revenue over $[t,T]$ from switching immediately, to the expected revenue if we delay the switch to a later time τ ($t \leq \tau \leq T$), then we can decide whether delaying the switch further is beneficial or not. The expected revenue values are $\Pi(t,n(t))$, and $E[p_B\big((N_B(\tau)-N_B(t)) \wedge n(t)\big)] + \Pi(\tau, n(\tau))$ for these two switching options. To compare these two expected values, we need a tool to measure the effect of the delay. Define the infinitesimal generator $\mathcal{G}$ with respect to the Poisson process $(t, N_B(t))$ for a uniformly bounded function $g(t,n)$ as

$$\mathcal{G}g(t,n) \;=\; \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[g(t+\Delta t, n - N_B(\Delta t)) - g(t,n)] = \frac{\partial g(t,n)}{\partial t} + \lambda_B[g(t,n-1) - g(t,n)].$$

Applying $\mathcal{G}$ to the function $\Pi(t,n(t))$ gives the immediate loss of revenue from single ticket sales if the switch to singles is delayed. Specifically, $\mathcal{G}\Pi(t,n(t)) = \frac{\partial \Pi(t,n(t))}{\partial t} + \lambda_B[\Pi(t,n(t)-1) - \Pi(t,n(t))]$, which is composed of two parts: $\frac{\partial \Pi(t,n(t))}{\partial t}$, which is the loss of revenue due to elapsed time, and $\lambda_B[\Pi(t,n(t)-1) - \Pi(t,n(t))]$, which is the loss of revenue due to the decrease in inventory to be sold as singles. But during the time when switching is delayed, the Poisson process for bundles $(t, N_B(t))$ is active, and it generates revenue at the rate $\mathcal{G}E\big[p_B\big((N_B(\tau)-N_B(t)) \wedge n(t)\big)\big] = \lambda_B p_B$. Therefore, the net marginal gain (or loss) for delaying the switch from bundles to singles at state $(t,n(t))$ is given by

$$\mathcal{G}\Pi(t,n(t)) + \lambda_B p_B = \frac{\partial \Pi(t,n(t))}{\partial t} + \lambda_B[\Pi(t,n(t)-1) - \Pi(t,n(t))] + \lambda_B p_B.$$

By Dynkin's Lemma (Rogers and Williams [20]), the following two expressions are martingales for any s $\geq t$:

$$\Pi(s,n(s)) - \Pi(t,n(t)) \;-\; \int_t^s \mathcal{G}\Pi(u,n(u))du, \tag{1}$$

$$p_B\big((N_B(s)-N_B(t)) \wedge n(t)\big) \;-\; \int_t^s \lambda_B p_B I_{\{n(u)>0\}} du, \tag{2}$$

where $I_{\{n(u)>0\}}$ is the indicator function. Since the expected value of these martingales at any time s is equal to their expected value at the starting time t, we have:

$$\Pi(s,n(s)) - \Pi(t,n(t)) = E\int_t^s \mathcal{G}\Pi(u,n(u))du, \tag{3}$$

$$E[p_B\big((N_B(s)-N_B(t)) \wedge n(t)\big)] = E\int_t^s \lambda_B p_B I_{\{n(u)>0\}} du. \tag{4}$$

By the optional sampling theorem (Karatzas and Shreve [14]), we can replace s in (3) and (4) with any stopping time $\tau \geq t$. Therefore, adding equations (3) and (4) for a stopping time τ, we get:

$$\begin{aligned} E[p_B\big((N_B(\tau) - N_B(t)) \wedge n(t)\big)] + \Pi(\tau, n(\tau)) \quad &- \quad \Pi(t, n(t)) \\ &= \quad E\int_t^\tau \big[\mathcal{G}\Pi(u, n(u)) + \lambda_B p_B I_{\{n(u)>0\}}\big]du. \end{aligned} \tag{5}$$

Note that the left-hand side of (5) is the expected revenue gained over $\Pi(t, n(t))$ by delaying the switch from t to τ, and we can quantify it by using $\mathcal{G}$, as shown in the right-hand side. Therefore, we know that delaying the switch to τ from t is beneficial if $E\int_t^\tau \big[\mathcal{G}\Pi(u, n(u)) + \lambda_B p_B I_{\{n(u)>0\}}\big]du > 0$.

Taking the supremum of both sides in (5) over all stopping times $t \leq \tau \leq T$, and defining

$$\widetilde{V}(t, n(t)) = \sup_{t\leq\tau\leq T} E\int_t^\tau \big[\mathcal{G}\Pi(u, n(u)) + \lambda_B p_B I_{\{n(u)>0\}}\big]du, \tag{6}$$

we get that $V(t, n(t)) = \Pi(t, n(t)) + \widetilde{V}(t, n(t))$. This implies that the optimal revenue over $[t, T]$ consists of two parts: the revenue from the immediate switch (selling single tickets until the end of horizon) and the additional revenue from delaying the switch further in time. Since $\widetilde{V}(t, n(t))$ is also given by

$$\widetilde{V}(t, n(t)) = \sup_{t\leq\tau\leq T} E[p_B\big((N_B(\tau) - N_B(t)) \wedge n(t)\big) + \Pi(\tau, n(\tau))] - \Pi(t, n(t)), \tag{7}$$

it is obvious that $\widetilde{V}(t, n(t)) \geq 0$ for any $0 \leq t \leq T$ and $0 \leq n(t) \leq M$. In particular, $\widetilde{V}(t, 0) = 0$ for all $0 \leq t \leq T$ and $\widetilde{V}(T, n(t)) = 0$ for all $0 \leq n(t) \leq M$. Moreover, equation (7) indicates that when $\widetilde{V}(t, n(t)) = 0$, delaying the switch further is not optimal, whereas $\widetilde{V}(t, n(t)) > 0$ implies a revenue potential from delaying the switch.

To compute $\widetilde{V}(t, n(t))$, we introduce the function $\overline{V}(t, n(t))$, which can be derived recursively and is identical to $\widetilde{V}(t, n(t))$ when a number of conditions are satisfied in Theorem 1.

Theorem 1. *Suppose there exists a function $\overline{V}(t, n(t))$ such that $\overline{V}(t, n(t))$ is continuous and differentiable with right continuous derivatives in [0,T] for each fixed $n(t)$. In addition, if $\overline{V}(t, n(t))$ satisfies:*

(i) $\overline{V}(t, n(t)) \geq 0, \quad 0 \leq t \leq T$ *and* $0 \leq n(t) \leq M$;

(ii) $\overline{V}(T, n(t)) = 0$ *for* $0 \leq n(t) \leq M$ *and* $\overline{V}(t, 0) = 0$ *for* $0 \leq t \leq T$;

(iii) $\overline{V}(t, n(t)) = 0 \Rightarrow \mathcal{G}(\overline{V} + \Pi)(t, n(t)) + \lambda_B p_B \leq 0$, $0 \leq t \leq T$ *and* $0 \leq n(t) \leq M$;

(iv) $\overline{V}(t, n(t)) > 0 \Rightarrow \mathcal{G}(\overline{V} + \Pi)(t, n(t)) + \lambda_B p_B = 0$, $0 \leq t \leq T$ *and* $0 \leq n(t) \leq M$;

then $\overline{V}(t,n(t)) = \widetilde{V}(t,n(t))$.

The first two conditions in the list are the non-negativity property and the boundary conditions of $\overline{V}$. Also, since $\widetilde{V}(t,n(t))$ determines whether it is optimal to switch immediately or not, the conditions for $\overline{V}(t,n(t))$ to be positive or zero are also crucial and are listed in conditions (*iii*) and (*iv*).

The proof for Theorem 1 is along the lines of the proof of Theorem 1 in Feng and Xiao [10] so is provided in Duran [6]. The function $\overline{V}(t,n(t))$ enables us to decide whether to delay the switch further than t is beneficial or not. The net marginal gain from delaying, $\mathcal{G}\Pi(t,n(t)) + \lambda_B p_B$, is the main term that defines the behavior of $\overline{V}(t,n(t))$, and this term will be addressed more closely in the following lemma. First, by noting that $E[(N_i(T) - N_i(t)) \wedge n(t)] = \sum_{k=1}^{n(t)} P[N_i(T) - N_i(t) \geq k]$, we can express $\Pi(t,n(t))$ for $n(t) \geq 1$ as

$$\Pi(t,n(t)) = p_1 \sum_{k=1}^{n(t)} P[N_1(T) - N_1(t) \geq k] + p_2 \sum_{k=1}^{n(t)} P[N_2(T) - N_2(t) \geq k]. \tag{8}$$

Lemma 2. *The net marginal gain from delaying for* $0 \leq t \leq T$ *can be written as*

$$\begin{aligned} \mathcal{G}\Pi(t,n(t)) + \lambda_B p_B = (r_B - r_1 - r_2) \quad &+ \quad p_1(\lambda_1 - \lambda_B) P[N_1(T) - N_1(t) \geq n(t)] \\ &+ \quad p_2(\lambda_2 - \lambda_B) P[N_2(T) - N_2(t) \geq n(t)]. \end{aligned}$$

See the Appendix for the proof. Note that when $\lambda_B > \lambda_i$ for $i = 1,2$, clearly $\mathcal{G}\Pi(t,n(t)) + \lambda_B p_B$ is increasing in t and n.

Although we have demonstrated the existence of the alternate function $\overline{V}(t,n(t))$, the issue of how to calculate it for any $(t,n(t))$ pairs still remains. From condition (*iv*), we know that $\mathcal{G}\overline{V}(t,n(t)) = -\mathcal{G}\Pi(t,n(t)) - \lambda_B p_B$ when $\overline{V}(t,n(t)) > 0$. Applying the infinitesimal generator $\mathcal{G}$ to $\overline{V}(t,n(t))$, we get the differential equation

$$\frac{\partial \overline{V}(t,n(t))}{\partial t} - \lambda_B \overline{V}(t,n(t)) = -[\lambda_B \overline{V}(t,n(t)-1) + \mathcal{G}\Pi(t,n(t)) + \lambda_B p_B], \tag{9}$$

which has the solution $\overline{V}(t,n(t)) = \displaystyle\int_t^T e^{-\lambda_B(s-t)}[\lambda_B \overline{V}(s,n(t)-1) + \mathcal{G}\Pi(s,n(t)) + \lambda_B p_B]ds$ provided that $\overline{V}(t,n(t)-1)$ is known. Since $\overline{V}(t,0) = 0$, all $\overline{V}(t,n(t))$ can be solved recursively. The formal procedure is given in the following theorem. We will prove that the $\overline{V}(t,n(t))$ that is determined by the proposed recursive procedure satisfies conditions (*i*)-(*iv*), and is thus equivalent to $\widetilde{V}(t,n(t))$. Moreover this procedure also determines the latest switching times $(x_{n(t)})$ for each possible unsold inventory level $n(t)$.

Theorem 3. *For all $1 \leq n(t) \leq M$ and $\lambda_B > \lambda_i$, for $i = 1, 2$, $\overline{V}(t, n(t))$ is recursively determined by*

$$\overline{V}(t,n(t)) = \begin{cases} \displaystyle\int_t^T L(s,n(t))e^{-\lambda_B(s-t)}ds & \text{if } t > x_{n(t)} \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

where

$$\begin{aligned} x_{n(t)} &= \inf\big\{0 \leq t \leq T : \int_t^T L(s,n(t))e^{-\lambda_B(s-t)}ds > 0\big\}, \\ L(t,n(t)) &= \mathcal{G}\Pi(t,n(t)) + \lambda_B p_B + \lambda_B \overline{V}(t,n(t)-1),\ 0 \leq t \leq T, \\ \overline{V}(t,0) &= 0,\ 0 \leq t \leq T. \end{aligned}$$

The proof for the Theorem is along the lines of the proof of Theorem 2 in Feng and Xiao [10] and is available in Duran [6]. What we have shown so far is that for any inventory level $n = 1, \ldots, M$, there exists a time x_n such that: $\overline{V}(t,n) > 0$ if $t > x_n$, and $\overline{V}(t,n) = 0$ if $t \leq x_n$. Therefore, if the system reaches remaining inventory level n at a time $t \leq x_n$, then it is optimal to switch immediately. On the other hand if it takes the system longer than x_n time units to reach remaining inventory level n, then it is optimal to delay the switch. Therefore, the x_n values can be interpreted as the latest switching time or the switching-time thresholds, when n items are unsold.

Moreover, while proving Theorem 3, we show that the switching-time thresholds $\{x_n\}$, $n = 1, \ldots, M$ are non-increasing in unsold inventory n. Intuitively, as the unsold inventory increases, it is beneficial for the team to delay the switch for more time in order to take advantage of the bundle sales longer.

4 Extensions

The dynamic switching problem that is considered so far assumes constant demand rates and a 2-event selling season. In this section, we relax these two assumptions.

4.1 ℓ-Performances ($\ell > 2$) During the Selling Period

When there are more than two events on sale, the only difference from the base case is the expression for the profit from the individual ticket sales over $[t, T]$ with $n(t)$ items available. It is given by

$$\Pi(t,n(t)) = \sum_{i=1}^{\ell} p_i E[(N_i(T) - N_i(t)) \wedge n(t)] = \sum_{i=1}^{\ell}\sum_{k=1}^{n(t)} p_i P[N_i(T) - N_i(t) \geq k].$$

It is easy to see that when $\mathcal{G}$ is applied to $\Pi(t, n(t))$ we obtain

$$\mathcal{G}\Pi(t, n(t)) + \lambda_B p_B = (r_B - \sum_{i=1}^{\ell} r_i) \quad + \quad \sum_{i=1}^{\ell} p_i(\lambda_i - \lambda_B) P[N_i(T) - N_i(t) \geq n(t)].$$

This expression is increasing in t and n, when $\lambda_i \leq \lambda_B$ for each $i = 1, \ldots, \ell$, and $r_B > \sum_{i=1}^{\ell} r_i$. By using the same analysis technique as in the base case, we obtain the following corollary.

Corollary 4. *If* $\lambda_i \leq \lambda_B$ *for each* $i = 1, \ldots, \ell$, *and* $r_B > \sum_{i=1}^{\ell} r_i$, *the switching times* $\{x_i\}$ $i = 1, \ldots, M$ *are non-increasing in the remaining inventory.*

4.2 Time-Dependent Demand Rates

So far we have assumed that the Poisson processes associated with the pre-determined prices have constant demand rates. However, it is also possible that the demand rates may change with the remaining time. For example, they may decrease with time, since demand can be related to seat quality. Incorporation of time-dependent demand rates into the formulation of the problem will enable us to indirectly model other aspects such as: *i*) substitution of products, where a decrease in the demand rate for one product may be due to the other product being on sale longer or *ii*) word-of-mouth effect, where demand rates increase with time due to the increase in publicity for an event.

We keep the problem setting the same as in Section 3 and take the Poisson processes as follows: $N_B(s)$, $0 \leq s \leq t$, with intensity $\lambda_B(t)$, $N_1(s)$, $0 \leq s \leq t$, with intensity $\lambda_1(t)$, and $N_2(s)$, $0 \leq s \leq t$, with intensity $\lambda_2(t)$. Furthermore let $r_B(t) = \lambda_B(t) p_B$ be the expected revenue rate from the bundled and $r_i(t) = \lambda_i(t) p_i$ be the expected revenue rate from the individual ticket sales at time t. The infinitesimal generator $\overline{\mathcal{G}}$ is defined as

$$\overline{\mathcal{G}} g(t, n) \quad = \quad \frac{\partial g(t, n)}{\partial t} + \lambda_B(t)[g(t, n-1) - g(t, n)].$$

Note that $\overline{\mathcal{G}}$ is similar to $\mathcal{G}$ but incorporates the dependence of the demand rate on t. Applying $\overline{\mathcal{G}}$ to $\Pi(t, n(t))$, we get

$$\begin{aligned} \overline{\mathcal{G}}\Pi(t, n(t)) + \lambda_B(t) p_B = [r_B(t) - r_1(t) - r_2(t)] \quad &+ \quad p_1(\lambda_1(t) - \lambda_B(t)) P[N_1(T) - N_1(t) \geq n(t)] \\ &+ \quad p_2(\lambda_2(t) - \lambda_B(t)) P[N_2(T) - N_2(t) \geq n(t)]. \end{aligned}$$

When for all t, $\lambda_i(t) \leq \lambda_B(t)$ for each $i = 1, \ldots, 2$, and $r_B(t) > r_1(t) + r_2(t)$, then the last expression is increasing in t and n. By using the same analysis technique as in the base case, we obtain the following corollary.

Corollary 5. *If for all* $0 \leq t \leq T$*,* $\lambda_i(t) \leq \lambda_B(t)$ *for each* $i = 1, \ldots, 2$*, and* $r_B(t) > r_1(t) + r_2(t)$*, the switching times* $\{x_i\}$ $i = 1, \ldots, M$ *are non-increasing in remaining inventory.*

5 Computational Experiments

In this section, we present computational analysis illustrating the connection between the problem parameters and the optimal switching times. To calculate the optimal switching times, we need to calculate the revenue potential from delaying the switch, $\overline{V}(t, n(t))$. After dividing the selling period into small time intervals with size δ, we calculate $\overline{V}(t, n(t))$ for all inventory levels and times, recursively starting from $\overline{V}(T - \delta, 1)$. The $\overline{V}(t, n(t))$ function can be estimated by

$$\overline{V}(t, n(t)) \quad \cong \quad (\overline{V} + \Pi)(t + \delta, n(t))e^{-\lambda_B \delta} + (1 - e^{-\lambda_B \delta})[p_B + (\overline{V} + \Pi)(t, n(t) - 1)] - \Pi(t, n(t)).$$

If the selling horizon T is divided into a large number of K intervals of length δ, we obtain

$$\overline{V}(k\delta, n(t)) \quad \cong \quad (\overline{V} + \Pi)((k+1)\delta, n(t))e^{-\lambda_B \delta} + (1 - e^{-\lambda_B \delta})[p_B + (\overline{V} + \Pi)(k\delta, n(t) - 1)] - \Pi(k\delta, n(t)).$$

Starting from the end of the selling horizon T, where $\overline{V}(T, \cdot) = 0$, the values for $\overline{V}$ can be calculated recursively (see the Appendix for details).

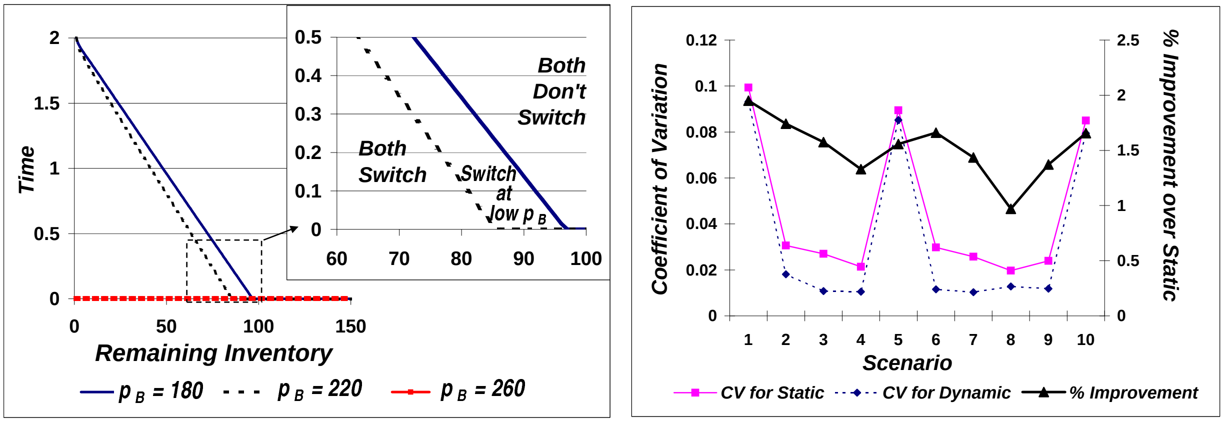
Consider a team with a 150-ticket stadium facing the problem of selling tickets to one high-demand and one low-demand game during a selling season that lasts 2 months. The demand rates for the games are 50 and 40 seats per month and the prices to be charged for these seats are \$200 and \$50 for the high and the low-demand game, respectively. If the seats are sold as a bundle with one high and one low-demand seat, the demand rate will be 100 seats per month for the bundle. Table 1 gives the ten optimal switching times calculated for the case when the bundle is sold with a price of \$220. As proved in Theorem 3, the optimal switching times are non-increasing in unsold inventory n.

Table 1: Selected Optimal Switching Times when $p_B = 220$

sales	73	72	71	70	69	68	67	66	65	64
remained seats ($n(t)$)	77	78	79	80	81	82	83	84	85	86
switch time ($x_{n(t)}$)	0.191	0.168	0.145	0.123	0.1	0.078	0.055	0.032	0.01	0

Consider the case when team has already sold 70 bundles (i.e., 80 seats remain). In this case, the optimal switch time can be computed to be equal to 0.123 months. If the team sold the 70 items before 0.123 months ($t < x_{80}$), it is optimal to switch before the 71^{st} sale arrives since there is no expected revenue potential from delaying the switch further ($\overline{V}(t, 80) = 0$). If the team sold the 70^{th} item in bundles after 0.123 months, then they should wait to switch. To illustrate how the switch times are used, it is also beneficial to consider the case when the optimal switch time is 0 with 86 seats left to sell in Table 1. Having zero switching times until the team sells 64 seats indicates that switching should be considered only after the 64^{th} sale.

Another area of interest is the behavior of the switching thresholds with the parameters of the model. Figure 1(a) shows the effect of different bundle prices on switching times. As the bundle price increases, the optimal switch threshold decreases for each inventory level (i.e., the time window for switching gets smaller), which is intuitive since the team tries to take advantage of high bundle prices by delaying the switch. If the bundle price is high enough, it may eliminate the switch option altogether, which occurs, for example when $p_B = 260$. Figure 1(a) also displays decision regions to illustrate how the switching decisions change for two different bundle prices ($p_B = 180$ and $p_B = 220$). Increasing the demand rates for the bundles has a similar effect as increasing bundle price, but the figure is left out for brevity.



(a) (b)

Figure 1: (a) Switching Thresholds for Different p_B; (b) Comparison of Dynamic Switch vs. Static

The final numerical experimentation demonstrates the impact on revenue of deciding the switch time dynamically instead of using a static switch time. The model parameters are the same as the ones that give the optimal switching times in Table 1. For a scenario, we create 100 random sample paths for the arrival of bundled and single ticket customers, and calculate the average revenue when the switch time is decided dynamically and statically for those paths; we repeat this scenario ten times to show the effect of different demand realizations. Figure 1(b) illustrates the percentage revenue improvement of dynamic switching over the static case can be between 1-2 % when the optimal static switching time (i.e., 1.2 months) is selected. Another important observation is that dynamic switch times can reduce the variation in expected profit. The revenue values are calculated for the same demand paths, therefore Figure 1(b) clearly illustrates that the usage of dynamic switching times improves the value and the predictability of the revenues that will be obtained. Also note that the potential for improvement from using dynamic switching thresholds tends to be higher when the variation in a scenario is higher.

6 Conclusions

In this paper, we have studied the problem of switching between selling bundles of tickets to selling individual tickets so as to maximize revenue over a selling season. This is a new problem in revenue management, which is motivated by discussions in the Sports and Entertainment industries, but it may have applications in other industries also. Important characteristics of the problem include that the bundle purchasers are sharing limited capacity with single-ticket purchasers, and when the switch is made the bundle splits into multiple simultaneous Poisson processes for single ticket demand. We also focus on the optimal *dynamic* switching time, where the switch time can depend on the state of the system.

We find that the structure of the problem yields an optimal policy that is intuitive and easy-to-implement. The optimal time to switch consists of a set of threshold pairs defined by the remaining inventory and the time left in the horizon. After each sale, the current time is compared to the time threshold for the corresponding remaining inventory to determine if the switch should be made immediately or not. The switching times balance the value of the bundle purchases over the single ticket purchases as well as the probability of future demand arrivals of each, and the switching times are non-increasing in unsold inventory. We also generalize these results in several ways including to ℓ events in the horizon or to a demand rate that depends on time, and we show that the same structural results hold. We find that dynamic decisions can improve revenues $1-2\%$ over the best static decision, which is comparable to improvements RM has brought to the airline industry, and that they can reduce profit variability.

There are several areas of research that could further benefit the dynamic switching problem. For instance, it would be interesting to explicitly consider how the results change when there are customer diversions between segments and how this compares to using time-dependent demand rates as a proxy. It could also be useful to study how to set the prices of bundles and single tickets and how decisions change when flexible bundles are considered.

7 Appendix

Proof of Lemma 2. Using the definition of differentiation, conditioning on $N_i(T)-N_i(t+h)$ and omitting the zero terms, we get

$$\begin{aligned}\frac{\partial P[N_i(T)-N_i(t)\geq k]}{\partial t} &= -\lim_{h\to 0}\frac{P[N_i(T)-N_i(t)\geq k]-P[N_i(T)-N_i(t+h)\geq k]}{h} \qquad (11)\\ &= -\lim_{h\to 0}\frac{\lambda_i h P(N_i(T)-N_i(t)=k-1)}{h}\\ &= -\lambda_i P(N_i(T)-N_i(t)=k-1). \qquad (12)\end{aligned}$$

Therefore, we have $\dfrac{\partial \sum_{k=1}^{n(t)} P[N_i(T)-N_i(t)\geq k]}{\partial t} = -\lambda_i P[N_i(T)-N_i(t)\leq n(t)-1]$. Using this equality we have

$$\begin{aligned}\mathcal{G}\Pi(t,n(t)) &= \frac{\partial \Pi(t,n(t))}{\partial t}+\lambda_B[\Pi(t,n(t)-1)-\Pi(t,n(t))]\\ &= -\lambda_1 p_1 P[N_1(T)-N_1(t)\leq n(t)-1]-\lambda_2 p_2 P[N_2(T)-N_2(t)\leq n(t)-1]\\ &\quad -\lambda_B p_1 P[N_1(T)-N_1(t)\geq n(t)]-\lambda_B p_2 P[N_2(T)-N_2(t)\geq n(t)]\\ &= -\lambda_1 p_1\big(1-P[N_1(T)-N_1(t)\geq n(t)]\big)-\lambda_2 p_2\big(1-P[N_2(T)-N_2(t)\geq n(t)]\big)\\ &\quad -\lambda_B p_1 P[N_1(T)-N_1(t)\geq n(t)]-\lambda_B p_2 P[N_2(T)-N_2(t)\geq n(t)]\\ &= -\lambda_1 p_1-\lambda_2 p_2+p_1(\lambda_1-\lambda_B)P[N_1(T)-N_1(t)\geq n(t)]\\ &+ \ p_2(\lambda_2-\lambda_B)P[N_2(T)-N_2(t)\geq n(t)].\end{aligned}$$

□

Details for the Algorithm for Calculating $\overline{V}$

The following algorithm guides computations from inventory level $n=1$ to M.

Algorithm

$$\begin{aligned}\textit{Let, } \Delta L(k\delta,n(t)) &= (\overline{V}+\Pi)((k+1)\delta,n(t))e^{-\lambda_B\delta}\\ &+ \ (1-e^{-\lambda_B\delta})[p_B+(\overline{V}+\Pi)(k\delta,n(t)-1)]-\Pi(k\delta,n(t)).\end{aligned}$$

- **Step 0**: Initialize $\overline{V}(T,\cdot)=\overline{V}(K\delta,\cdot)=0$ for all inventory levels. Set $n(t)=1$ and $k=(K-1)$.
- **Step 1**: Calculate $\Delta L(k\delta,n(t))$.

- **Step 2**: Set $\overline{V}(k\delta, n(t)) = (\Delta L(k\delta, n(t)))^+$ and $k = k - 1$.
  - if $k \neq -1$ and $\overline{V}(k\delta, n(t)) \geq 0$, go to Step 1;
  - otherwise set $\overline{V}(j\delta, n(t)) = 0$ for all $j < k - 1$ and $n = n + 1$.

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