Supplemental Material for Dynamic Switching Times for Season and Single Tickets in Sports and Entertainment

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June 2007

Details for the Derivation of the Infinitesimal Generator $\mathcal{G}$ Function

$$\begin{aligned}
\mathcal{G}g(t,n) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}E[g(t+\Delta t, n-N_B(\Delta t))-g(t,n)]\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\sum_{k=0}^{\infty}[g(t+\Delta t,(n-k)^+)-g(t,n)]\frac{(\lambda_B\Delta t)^k}{k!}e^{-\lambda_B\Delta t}\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\big[(g(t+\Delta t,n)-g(t,n))(1-\lambda_B\Delta t)+(g(t+\Delta t,n-1)-g(t,n))\lambda_B\Delta t\big]\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}(g(t+\Delta t,n)-g(t,n))+\lim_{\Delta t\to 0}\lambda_B(g(t+\Delta t,n-1)-g(t+\Delta t,n))\\
&= \frac{\partial g(t,n)}{\partial t}+\lambda_B[g(t,n-1)-g(t,n)].
\end{aligned}$$

Details for the Approximation of the $\overline{V}$ Function

The details of the approximation using discrete time intervals (see also [1]) are given below. For any $1\le n(t)\le M$, and $x_{n(t)}<t<T$ with some $\delta>0$ such that $t+\delta\le T$, we have

$$\begin{aligned}
\overline{V}(t,n(t)) &= \int_t^T L(u,n(t))e^{-\lambda_B(u-t)}du\\
&= \int_{t+\delta}^T L(u,n(t))e^{-\lambda_B(u-t)}du+\int_t^{t+\delta} L(u,n(t))e^{-\lambda_B(u-t)}du\\
&= \int_{t+\delta}^T L(u,n(t))e^{-\lambda_B(u-(t+\delta))}e^{-\lambda_B\delta}du+\int_t^{t+\delta} L(u,n(t))e^{-\lambda_B(u-t)}du\\
&\cong \overline{V}(t+\delta,n(t))e^{-\lambda_B\delta}+\int_t^{t+\delta} L(u,n(t))e^{-\lambda_B(u-t)}du\\
&\cong \overline{V}(t+\delta,n(t))e^{-\lambda_B\delta}\\
&+ \int_t^{t+\delta}\big[\mathcal{G}\Pi(u,n(t))+\lambda_B p_B+\lambda_B\overline{V}(u,n(t)-1)\big]e^{-\lambda_B(u-t)}du\\
&\cong \overline{V}(t+\delta,n(t))e^{-\lambda_B\delta}+(1-e^{-\lambda_B\delta})p_B+(1-e^{-\lambda_B\delta})\overline{V}(t,n(t)-1)\\
&+ (1-e^{-\lambda_B\delta})[\Pi(t,n(t)-1)-\Pi(t,n(t))]+e^{-\lambda_B\delta}[\Pi(t+\delta,n(t))-\Pi(t,n(t))].
\end{aligned}$$

Therefore $\overline{V}(t, n(t))$ can be estimated by

$$\overline{V}(t, n(t)) \;\cong\; (\overline{V} + \Pi)(t + \delta, n(t))e^{-\lambda_B \delta} + (1 - e^{-\lambda_B \delta})[p_B + (\overline{V} + \Pi)(t, n(t) - 1)] - \Pi(t, n(t)).$$

If the selling horizon T is divided into a large number K of intervals of length δ, we obtain

$$\overline{V}(k\delta, n(t)) \;\cong\; (\overline{V} + \Pi)((k + 1)\delta, n(t))e^{-\lambda_B \delta} + (1 - e^{-\lambda_B \delta})[p_B + (\overline{V} + \Pi)(k\delta, n(t) - 1)] - \Pi(k\delta, n(t)).$$

Starting from the end of the selling horizon T, where $\overline{V}(T, \cdot) = 0$, the following algorithm guides computations from inventory level $n = 1$ to M.

Proof of Theorem 1. $\widetilde{V}$ is defined as:

$$\widetilde{V}(t, n(t)) = \sup_{t \le \tau \le T} E \int_t^{\tau} \left[\mathcal{G}\Pi(u, n(u)) + \lambda_B p_B I_{\{n(u) > 0\}}\right] du. \tag{1}$$

Assume that there exists a function satisfying the conditions in the theorem. We will show that $\overline{V}$ is equal to $\widetilde{V}$. For $s \ge t$, let

$$\begin{aligned} m(s) = \overline{V}(s, [n(t) - N_B(s) + N_B(t)]^+) \;&-\; \overline{V}(t, n(t)) \\ &-\; \int_t^s \mathcal{G}\overline{V}(u, [n(t) - N_B(u) + N_B(t)]^+) du. \end{aligned}$$

$m(s)$ is a martingale by Dynkin's Lemma, and since the expected value of this martingale at any time s is equal to its expected value at the starting time t, we have $Em(s) = 0$. Further, by the optional sampling theorem, for any stopping time $\tau \ge t$ we have

$$E\left[\overline{V}(\tau, [n(t) - N_B(\tau) + N_B(t)]^+)\right] - E\int_t^{\tau} \mathcal{G}\overline{V}(u, [n(t) - N_B(u) + N_B(t)]^+) du = \overline{V}(t, n(t)) \tag{2}$$

$$\begin{aligned} & E\left[\overline{V}(\tau, [n(t) - N_B(\tau) + N_B(t)]^+)\right] \\ - \;& E\int_t^{\tau} \left[\mathcal{G}(\overline{V} + \Pi)(u, [n(t) - N_B(u) + N_B(t)]^+) + \lambda_B p_B I_{\{N_B(u) - N_B(t)) < n(t)\}}\right] du \\ = \;& \overline{V}(t, n(t)) - E\int_t^{\tau} \left[\mathcal{G}\Pi(u, [n(t) - N_B(u) + N_B(t)]^+) + \lambda_B p_B I_{\{N_B(u) - N_B(t)) < n(t)\}}\right] du. \end{aligned} \tag{3}$$

If we subtract $E\int_t^{\tau}\left[\mathcal{G}\Pi(u, [n(t) - N_B(u) + N_B(t)]^+) + \lambda_B p_B I_{\{N_B(u) - N_B(t)) < n(t)\}}\right] du$ from both sides of (2), the left-hand side of the resulting term, given by (3), is always positive by conditions (i), (iii) and (iv). Therefore,

$$\overline{V}(t, n(t)) \ge E\int_t^{\tau} \left[\mathcal{G}\Pi(u, [n(t) - N_B(u) + N_B(t)]^+) + \lambda_B p_B I_{\{N_B(u) - N_B(t)) < n(t)\}}\right] du.$$

Since $\overline{V}(t, n(t))$ is greater than or equal to each term in the right-hand side of equation (1) for any τ, it is also greater than or equal to the supremum over all τ, which is $\widetilde{V}(t, n(t))$ in equation (1). Hence, we conclude that $\overline{V}(t, n(t)) \ge \widetilde{V}(t, n(t))$ for any stopping time $\tau \ge t$.

To prove that $\overline{V}(t,n(t)) \leq \widetilde{V}(t,n(t))$, we will define a specific stopping time. Let σ be defined as $\sigma = \inf\big\{t \leq s \leq T : \overline{V}(s,[n(t)-N_B(s)+N_B(t)]^+) = 0\big\}$. Note that σ is well-defined because $\overline{V}(T,\cdot) = 0$. Replacing τ in equation (3) with σ, we obtain

$$\begin{aligned} & E\big[\overline{V}(\sigma,[n(t)-N_B(\sigma)+N_B(t)]^+)\big] \\ - \;& E\int_t^\sigma \big[\mathcal{G}(\overline{V}+\Pi)(u,[n(t)-N_B(u)+N_B(t)]^+) + \lambda_B p_B I_{\{N_B(u)-N_B(t))<n(t)\}}\big]du \\ = \;& \overline{V}(t,n(t)) - E\int_t^\sigma \big[\mathcal{G}\Pi(u,[n(t)-N_B(u)+N_B(t)]^+) + \lambda_B p_B I_{\{N_B(u)-N_B(t))<n(t)\}}\big]du. \end{aligned} \tag{4}$$

The definition of σ implies that $\overline{V}(\sigma,[n(t)-N_B(\sigma)+N_B(t)]^+) = 0$, and the definition of σ and condition (iv) together imply that $\mathcal{G}(\Pi)(u,[n(t)-N_B(u)+N_B(t)]^+) + \lambda_B p_B = 0$ for all $u \in [t,\sigma]$. Therefore the left-hand side of (4) is zero and we have

$$\begin{aligned} \overline{V}(t,n(t)) &= E\int_t^\sigma \big[\mathcal{G}\Pi(u,[n(t)-N_B(u)+N_B(t)]^+) + \lambda_B p_B I_{\{N_B(u)-N_B(t))<n(t)\}}\big]du \\ &\leq \widetilde{V}(t,n(t)). \end{aligned}$$

The inequality follows from the fact that the left-hand side of the inequality is the right-hand side of equation (1) for a specific stopping time, and $\widetilde{V}(t,n(t))$ is the supremum over all stopping times τ in that equation. Hence, $\overline{V}(t,n(t)) = \widetilde{V}(t,n(t))$. □

Obtaining $\overline{V}$ from the Differential Equation

The following theorem is used for the general solution of a first order linear inhomogeneous differential equation.

Theorem A.1. *If $g(t)$ is the solution of the differential equation*

$$g'(t) + \lambda g(t) = h(t), \quad 0 \leq t \leq T$$

where λ is a real number, $h(t)$ is a continuous function, then

$$g(t) = g(T)e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)}h(s)ds$$

Proof. Define $G(t) = e^{\lambda t}g(t)$. Then $g(t) = G(t)e^{-\lambda t}$ and $g'(t) = G'(t)e^{-\lambda t} - \lambda G(t)e^{-\lambda t}$. Consequently, $g'(t) + \lambda g(t) = G'(t)e^{-\lambda t}$. Then $h(t) = G'(t)e^{-\lambda t}$ $(*)$, and $(*)$ has solution

$$G(t) = G(T) - \int_t^T e^{\lambda s}h(s)ds$$

then, $g(t) = g(T)e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)}h(s)ds$. □

Therefore the solution of the differential equation:

$$\frac{\partial \overline{V}(t,n(t))}{\partial t} - \lambda_B \overline{V}(t,n(t)) = -[\lambda_B \overline{V}(t,n(t)-1) + \mathcal{G}\Pi(t,n(t)) + \lambda_B p_B], \tag{5}$$

is $\overline{V}(t,n(t)) = \int_t^T e^{-\lambda_B(s-t)}[\lambda_B \overline{V}(s,n(t)-1) + \mathcal{G}\Pi(s,n(t)) + \lambda_B p_B]ds = \int_t^T e^{-\lambda_B(s-t)} L(s,n(t))ds$, since $\overline{V}(T,\cdot) = 0$.

Proof of Theorem 3. The proof will be done by induction on $n(t)$. In the theorem $\overline{V}$ is defined as:

$$\overline{V}(t,n(t)) = \begin{cases} \int_t^T L(s,n(t))e^{-\lambda_B(s-t)}ds & if\ t > x_{n(t)} \\ 0 & otherwise. \end{cases} \tag{6}$$

When $n(t) = 1$, $\overline{V}(t,n(t)-1) = 0$, and we have $L(t,1) = \mathcal{G}\Pi(t,1) + \lambda_B p_B$, which is an increasing function in t since $\mathcal{G}\Pi(t,1)$ is an increasing function in t. We claim that for $t \le x_1$ the following holds:

$$L(t,1) \;=\; \mathcal{G}\Pi(t,1) + \lambda_B p_B \le \mathcal{G}\Pi(x_1,1) + \lambda_B p_B \le 0.$$

The first inequality is a consequence of the increasing property of $\mathcal{G}\Pi(t,1)$ in t. The second inequality follows from the fact that if $\mathcal{G}\Pi(x_1,1) + \lambda_B p_B > 0$ then $\int_{x_1}^T L(s,n(t))e^{-\lambda_B(s-t)}ds > 0$, which contradicts the definition of x_1. Hence, for $t \le x_1$ (or $\overline{V}(t,1) = 0$ by the definition of $\overline{V}$)

$$\begin{aligned} \mathcal{G}(\overline{V}+\Pi)(t,1) + \lambda_B p_B &= \frac{\partial \overline{V}(t,1)}{\partial t} + \lambda_B[\overline{V}(t,0) - \overline{V}(t,1)] + \mathcal{G}\Pi(t,1) + \lambda_B p_B \\ &= \mathcal{G}\Pi(t,1) + \lambda_B p_B = L(t,1) \le 0. \end{aligned}$$

Thus, condition (iii) is satisfied when $n(t) = 1$ and $t \le x_1$ (or $\overline{V}(t,1) = 0$).

When $t > x_1$ (or $\overline{V}(t,1) > 0$) we have that

$$\begin{aligned} \mathcal{G}(\overline{V}+\Pi)(t,1) + \lambda_B p_B &= \frac{\partial \overline{V}(t,1)}{\partial t} + \lambda_B[\overline{V}(t,0) - \overline{V}(t,1)] + \mathcal{G}\Pi(t,1) + \lambda_B p_B \\ &= \frac{\partial \overline{V}(t,1)}{\partial t} - \lambda_B \overline{V}(t,1) + \mathcal{G}\Pi(t,1) + \lambda_B p_B. \end{aligned} \tag{7}$$

By the definition of $\overline{V}(t,n(t))$, we have $\overline{V}(t,1) = \int_t^T L(s,1)e^{-\lambda_B(s-t)}ds$. Taking the derivative with respect to t, we get

$$\frac{\partial \overline{V}(t,1)}{\partial t} = \int_t^T \lambda_B L(s,1)e^{-\lambda_B(s-t)}ds - L(t,1) = \lambda_B \overline{V}(t,1) - \mathcal{G}\Pi(t,1) - \lambda_B p_B. \tag{8}$$

Substituting (8) into (7), we get $\mathcal{G}(\overline{V}+\Pi)(t,1) + \lambda_B p_B = 0$. $\overline{V}(t,1) > 0$. Therefore, condition (iv) is satisfied when $n(t) = 1$. Moreover, we have $\overline{V}(t,1) \ge \overline{V}(t,0) = 0$ by the definition of x_1 (there exists a time t such that $\overline{V}(t,1) > 0$ if $x_1 > 0$).

Now assume that the following statements hold for $n(t) \leq k < M$: there exist k time thresholds with $T \geq x_1 \geq \cdots \geq x_k \geq 0$ such that $\overline{V}(t, n(t))$ is derived from equation (6) and satisfies conditions (i)-(iv), and the inequality $\overline{V}(t, n(t)) \geq \overline{V}(t, n(t) - 1)$ holds for $n(t) = 1 \ldots k$.

For $n(t) = k + 1$ we have that

$$L(t, k+1) = \mathcal{G}\Pi(t, k+1) + \lambda_B p_B + \lambda_B \overline{V}(t, k) \geq \mathcal{G}\Pi(t, k) + \lambda_B p_B + \lambda_B \overline{V}(t, k-1) = L(t, k),$$

since $\mathcal{G}\Pi(t, k)$ and $\overline{V}(t, k)$ are increasing in k by the induction assumption. This implies that

$$\int_t^T L(s, k+1) e^{-\lambda_B (s-t)} ds \geq \int_t^T L(s, k) e^{-\lambda_B (s-t)} ds.$$

Together with equation (6), this implies $\overline{V}(t, k+1) \geq \overline{V}(t, k)$ and $x_k \geq x_{k+1}$.

For $t \leq x_{k+1}$ (or $\overline{V}(t, k+1) = 0$),

$$\begin{aligned}
\mathcal{G}(\overline{V} + \Pi)(t, k+1) + \lambda_B p_B &= \frac{\partial \overline{V}(t, k+1)}{\partial t} + \lambda_B[\overline{V}(t, k) - \overline{V}(t, k+1)] + \mathcal{G}\Pi(t, k+1) + \lambda_B p_B \\
&= \mathcal{G}\Pi(t, k+1) + \lambda_B p_B + \lambda_B \overline{V}(t, k) = L(t, k+1) \\
&\leq L(x_{k+1}, k+1) \leq 0.
\end{aligned}$$

Note that $\overline{V}(t, k) = \overline{V}(t, k+1) = 0$ since $t \leq x_{k+1} \leq x_k$. The first inequality follows from $\mathcal{G}\Pi(t, k+1)$ being increasing in t, and the second inequality follows from the fact that if $L(x_{k+1}, k+1) > 0$ then this will contradict the definition of x_{k+1}. Therefore, condition (iii) is satisfied, when $t \leq x_{k+1}$ (or $\overline{V}(t, k+1) = 0$).

For $t > x_{k+1}$ (or $\overline{V}(t, k+1) > 0$),

$$\begin{aligned}
\mathcal{G}(\overline{V} \quad &+ \quad \Pi)(t, k+1) + \lambda_B p_B \\
&= \frac{\partial \overline{V}(t, k+1)}{\partial t} + \lambda_B[\overline{V}(t, k) - \overline{V}(t, k+1)] + \mathcal{G}\Pi(t, k+1) + \lambda_B p_B \\
&= -L(t, k+1) + \lambda_B \overline{V}(t, k+1) + \lambda_B[\overline{V}(t, k) - \overline{V}(t, k+1)] + \mathcal{G}\Pi(t, k+1) + \lambda_B p_B = 0.
\end{aligned}$$

Therefore condition (iv) is satisfied when $t > x_{k+1}$ (or $\overline{V}(t, k+1) = 0$).

For $n(t) = k + 1$ we showed that conditions (i)-(iv) hold. Thus the function $\overline{V}(t, k)$ that is determined by the proposed procedure, is equal to $\widetilde{V}(t, k)$. Further the switching time thresholds (x_n) are monotonically non-increasing in n. $\square$

References

[1] Y. Feng. *Continuous-time Models in Perishable Asset Revenue Management.* PhD thesis, Columbia University, 1994.