Supplemental Appendix for "Facilitating Demand Risk-Sharing with the Percent Deviation Contract"

Proof of Lemma 1. We will define the three realizations of (2) as follows:

$$\begin{aligned}\Pi^S_{A.I.} &= w\left[\int_0^{t_1+M} xf(x)dx + (t_1+M)\left(1-F(t_1+M)\right)\right] + p\int_0^{t_1+M}\left(t_1+M-x\right)f(x)dx + \\ &\quad v\int_0^{t_1}(t_1-x)f(x)dx - c_1t_1 - c_2\left[\int_{t_1}^{t_1+M}(x-t_1)f(x)dx + M\left(1-F(t_1+M)\right)\right] - \\ &\quad \alpha\int_{t_1+M}^{\infty}(x-t_1-M)f(x)dx \quad (27)\\ \Pi^S_{A.II.} &= w\left[\int_0^{t_1+M} xf(x)dx + (t_1+M)\left(1-F(t_1+M)\right)\right] + p\int_0^{(1-d)q_1}\left((1-d)q_1-x\right)f(x)dx + \\ &\quad v\int_0^{t_1}(t_1-x)f(x)dx - c_1t_1 - c_2\left[\int_{t_1}^{t_1+M}(x-t_1)f(x)dx + M\left(1-F(t_1+M)\right)\right] - \\ &\quad \alpha\int_{t_1+M}^{\infty}(x-t_1-M)f(x)dx \quad (28)\\ \Pi^S_{A.III.} &= w\left[\int_0^{t_1+M} xf(x)dx + (t_1+M)\left(1-F(t_1+M)\right)\right] + p\int_0^{(1-d)q_1}\left((1-d)q_1-x\right)f(x)dx + \\ &\quad p\left[\int_{(1+d)q_1}^{t_1+M}(x-(1+d)q_1)f(x)dx + \left(t_1+M-(1+d)q_1\right)\left(1-F(t_1+M)\right)\right] + \\ &\quad v\int_0^{t_1}(t_1-x)f(x)dx - c_1t_1 - c_2\left[\int_{t_1}^{t_1+M}(x-t_1)f(x)dx + M\left(1-F(t_1+M)\right)\right] - \\ &\quad \alpha\int_{t_1+M}^{\infty}(x-t_1-M)f(x)dx. \quad (29)\end{aligned}$$

The second derivative of (27) taken with respect to t_1 is $(p+c_2-w-\alpha)f(t_1+M)-(c_2-v)f(t_1)$, which is negative for all values of t_1 if $w+\alpha > p+c_2$ based on the parameter conditions of scenario A. The second derivative of (28) is $(c_2-w-\alpha)f(t_1+M)-(c_2-v)f(t_1)$, and the second derivative of (29) is $(c_2-w-\alpha-p)f(t_1+M)-(c_2-v)f(t_1)$. Both of these expressions are negative for all values of t_1 without the extra condition. ■

Proof of Lemma 2. This case can easily be compared with the centralized case C.II. in which the centralized firm also does not expedite. The total expected supply chain profit for the voluntary compliance case is

$$\Pi^{SC}_{B.} = r\left[\int_0^{t_1} xf(x)dx + t_1(1-F(t_1))\right] + v\int_0^{t_1}(t_1-x)f(x)dx - c_1t_1 - \beta\int_{t_1}^{\infty}(x-t_1)f(x)dx. \quad (30)$$

Comparing (30) with the centralized supply chain profit in (20), it is easily seen that the two profits will be equal if the t_1 decisions are equal, which is accomplished if $\frac{w+\alpha-c_1+p}{w+\alpha-v+p} = \frac{r+\beta-c_1}{r+\beta-v}$. Simplifying this equality yields the channel coordinating condition. ■

Proof of Lemma 3. This result follows directly from Lemma 5. ■

Proof of Lemma 4. Assume that this relationship is true. Since the piecewise functions are concave from Lemma 1, $t_1^{A.III.}$ is the single maximum of $\Pi_{A.III.}^S$, and $\Pi_{A.III.}^S(t_1)$ is decreasing for values of $t_1 > t_1^{A.III.}$. Consequently, $\Pi_{A.III.}^S(t_1^{A.III.}) > \Pi_{A.III.}^S((1+d)q_1 - M) > \Pi_{A.III.}^S(t_1^{A.II.})$. Since $\Pi_{A.II.}^S((1+d)q_1 - M) = \Pi_{A.III.}^S((1+d)q_1 - M)$ from Observation 1, we have $\Pi_{A.III.}^S(t_1^{A.III.}) > \Pi_{A.II.}^S((1+d)q_1 - M) > \Pi_{A.III.}^S(t_1^{A.II.})$. Observation 3 states that $\Pi_{A.III.}^S(t_1^{A.II.}) > \Pi_{A.II.}^S(t_1^{A.II.})$, which implies $\Pi_{A.III.}^S(t_1^{A.III.}) > \Pi_{A.II.}^S((1+d)q_1 - M) > \Pi_{A.III.}^S(t_1^{A.II.}) > \Pi_{A.II.}^S(t_1^{A.II.})$. The statement $\Pi_{A.II.}^S((1+d)q_1 - M) > \Pi_{A.II.}^S(t_1^{A.II.})$ contradicts the result that $t_1^{A.II.}$ maximizes $\Pi_{A.II.}^S$. ■

Lemma 5 *If $w + \alpha > p + c_2$, then $t_1^{A.I.} \geq t_1^{A.II.}$.*

Proof. Suppose, on the contrary, $t_1^{A.I.} < t_1^{A.II.}$, which implies that $F(t_1^{A.I.} + M) \leq F(t_1^{A.II.} + M)$. Substituting the values given in (3) and (4), we have

$$\begin{aligned} \frac{w + \alpha - c_1 - (c_2 - v)F(t_1^{A.I.})}{w + \alpha - c_2 - p} &\leq \frac{w + \alpha - c_1 - (c_2 - v)F(t_1^{A.II.})}{w + \alpha - c_2} \\ (c_2 - v)(w + \alpha - c_2)\left[F(t_1^{A.II.}) - F(t_1^{A.I.})\right] &\leq -p\left(w + \alpha - c_1 - (c_2 - v)F(t_1^{A.II.})\right). \end{aligned} \quad (31)$$

The left side of (31) is positive, and the right side is negative since the numerator in (4) must be positive at $t_1^{A.II.}$. (The denominator is positive due to the parameter relationship defining case A.) This leads to a contradiction. ■

Lemma 6 *If $w + \alpha > p + v$, then $t_1^{I.B.II.} \leq \min\left\{t_1^{B.I.}, t_1^{B.III.}\right\}$.*

Proof. We follow the same contradiction procedure as in the proof of Lemma 5. First, assume $t_1^{B.II.} > t_1^{B.I.}$, which implies that $F\left(t_1^{B.II.}\right) \geq F\left(t_1^{B.I.}\right)$. Further substitution yields

$$\begin{aligned} \frac{w + \alpha - c_1}{w + \alpha - v} &\geq \frac{w + \alpha - c_1}{w + \alpha - v - p} \\ -p(w + \alpha - c_1) &\geq 0, \end{aligned}$$

which leads to a contradiction since $w + \alpha - c_1 \geq 0$.

Similarly, we assume that $t_1^{B.II.} > t_1^{B.III.}$; consequently, $F\left(t_1^{B.II.}\right) \geq F\left(t_1^{B.III.}\right)$. We obtain the following contradiction through substitution.

$$\frac{w+\alpha-c_1}{w+\alpha-v} \geq \frac{w+\alpha-c_1+p}{w+\alpha-v+p} \Rightarrow v \geq c_1,$$

which is a contradiction since $c_1 > v$. ■

Proof of Lemma 7. We will define the four functions resulting from (7) as follows:

$$\begin{aligned}
\Pi_{A.I.}^{B} &= (r-w)\left[\int_0^{t_1^*(q_1)+M} xf(x)dx + \left(t_1^*(q_1)+M\right)\left(1-F\left(t_1^*(q_1)\right)\right)\right] - \\
&\quad p\int_0^{t_1^*(q_1)} \left(t_1^*(q_1)+M-x\right)f(x)dx + (\alpha-\beta)\int_{t_1^*(q_1)}^{\infty} \left(x-t_1^*(q_1)-M\right)f(x)dx \quad (32)\\
\Pi_{A.II.}^{B} &= (r-w)\left[\int_0^{t_1^*(q_1)+M} xf(x)dx + \left(t_1^*(q_1)+M\right)\left(1-F\left(t_1^*(q_1)\right)\right)\right] - \\
&\quad p\int_0^{(1-d)q_1} \left((1-d)q_1-x\right)f(x)dx + (\alpha-\beta)\int_{t_1^*(q_1)}^{\infty} \left(x-t_1^*(q_1)-M\right)f(x)dx \quad (33)\\
\Pi_{A.III.}^{B} &= (r-w)\left[\int_0^{t_1^*(q_1)+M} xf(x)dx + \left(t_1^*(q_1)+M\right)\left(1-F\left(t_1^*(q_1)\right)\right)\right] - \\
&\quad p\int_0^{(1-d)q_1} \left((1-d)q_1-x\right)f(x)dx + (\alpha-\beta)\int_{t_1^*(q_1)}^{\infty} \left(x-t_1^*(q_1)-M\right)f(x)dx - \quad (34)\\
&\quad p\left[\int_{(1+d)q_1}^{t_1^*(q_1)+M} (x-(1+d)q_1)f(x)dx + \left(t_1^*(q_1)+M-(1+d)q_1\right)\left(1-F\left(t_1^*(q_1)+M\right)\right)\right].\\
\Pi_{A.IV.}^{B} &= (r-w)\left[\int_0^{(1-d)q_1} xf(x)dx + (1-d)q_1\left(1-F\left((1-d)q_1\right)\right)\right] - \\
&\quad p\int_0^{(1-d)q_1} \left((1-d)q_1-x\right)f(x)dx + (\alpha-\beta)\int_{(1-d)q_1}^{\infty} \left(x-(1-d)q_1\right)f(x)dx \quad (35)
\end{aligned}$$

The second derivative of (32) is zero, so this component function is concave. The second derivative of (33) equals $-p(1-d)^2 f((1-d)q_1)$, which is negative for all values of q_1; thus, this function is concave. Similarly, the second derivative of (34) is $-p(1-d)^2 f((1-d)q_1) - p(1+d)^2 f((1+d)q_1)$, which is negative for all q_1. The second derivative of (35) is $-(r-w-\alpha+\beta)(1-d)^2 f((1-d)q_1) - p(1-d)^2 f((1-d)q_1)$, which is also negative for all q_1, making this function concave as well. ■

Proof of Theorem 2. The following lemma establishes the piecewise-concavity of (7) with respect to the four realizations of the buyer's expected profit function.

Lemma 7 *The buyer's expected profit function realizations resulting from (7) are concave.*

Since the buyer's four individual profit function realizations are concave from Lemma 7, we can use the KKT conditions to solve for the optimal q_1 for each function over the region of q_1 values where the function is valid, as defined in 1.

We first maximize (32) over the region $q_1 \geq \frac{t_1^{A.I.}+M}{1-d}$. Since (32) is not dependent on q_1, any value $q_1^{A.I.} \equiv \left\{q : q \geq \frac{t_1^{I.A.I.}+M}{1-d}\right\}$ is optimal.

We consider (33) over the region $q_1 \leq \frac{t_1^{A.II.}+M}{1-d} \bigcap q_1 \geq \frac{t_1^{A.II.}+M}{1+d} \bigcap q_1 \geq \frac{t_1^{A.III.}+M}{1+d}$. Taking the partial derivative and setting it equal to zero yields $p(1-d)F((1-d)q_1) = 0$. Since only the lower deviation penalty exists in this profit function realization, the buyer wants to make her initial order estimate as small as possible to avoid paying the penalty. Consequently, $q_1^* = q_1^{A.II.} \equiv \max\left\{\frac{F^{-1}(0)}{1-d}, \frac{\max\left\{t_1^{A.II.}, t_1^{A.III.}\right\}+M}{1+d}\right\}$.

We want to maximize (34) over the region $q_1 \leq \frac{t_1^{A.II.}+M}{1+d} \bigcap q_1 \leq \frac{t_1^{A.III.}+M}{1+d}$. First order optimality conditions yield $q_1^* = q_1^{A.III.} \equiv \{q : (1+d) = (1-d)F((1-d)q) + (1+d)F((1+d)q)\}$, which is feasible if it is smaller than $\frac{\min\left\{t_1^{A.II.}, t_1^{A.III.}\right\}+M}{1+d}$.

Finally, we maximize (35) over the region $q_1 \geq \frac{t_1^{A.II.}+M}{1-d} \bigcap q_1 \leq \frac{t_1^{A.I.}+M}{1-d} \bigcap q_1 \geq \frac{t_1^{A.III.}+M}{1+d}$. The first order conditions give us $q_1^* = q_1^{A.IV.} \equiv \left\{q : F((1-d)q_1) = \frac{r-w-\alpha+\beta}{r-w-\alpha+\beta+p}\right\}$, which is feasible if $\max\left\{\frac{t_1^{A.II.}+M}{1-d}, \frac{t_1^{A.III.}+M}{1+d}\right\} \leq q_1^{\beta} \leq \frac{t_1^{A.I.}+M}{1-d}$. ∎

Proof of Theorem 3. The proof of this result follows the same logic as that of Theorem 1, utilizing the results from Lemma 6. ∎