1. An umbrella company has estimated its monthly demand for the next year, with \(d_i\) being the estimated demand for each month \(i\). Each month of the year, the company can produce umbrellas. The cost per umbrella produced in month \(i\) is \(p_i\). (Because of the varying seasonal costs of raw materials, \(p_i\) might not be the same from month to month.)

The company must always meet its monthly demand. However, it can also overproduce; that is, it can produce more than is necessary to meet demand, and then hold the extra umbrellas in inventory. Holding inventory costs \(h_i\) dollars per umbrella in month \(i\).

At the beginning of the year, heading into the first month, the company has \(I_0\) umbrellas already in inventory. They would like to end the year with the same number of umbrellas in inventory.

Suppose there was a limit of \(u_i\) on the amount that could be produced by regular-time workers in month \(i\), but that they could exceed that limit by giving workers overtime pay. The first \(u_i\) umbrellas could be produced at the standard cost \(p_i\) each, and an additional \(v_i\) (or fewer) umbrellas could be produced at the overtime rate of \(o_i\) per umbrella, with \(o_i > p_i\).

(a) Draw this scenario as a network problem. If it is a special type of network problem, specify what type it is.
(b) Specify the corresponding linear program.

2. The Department of Transportation (DOT) is building a new highway. The highway goes through a hilly region, and instead of having the road go up and down repeatedly, the DOT would like to have the road ascend steadily. The figure below shows the current landscape and the proposed road.

In order to give the landscape a constant slope, dirt must be removed from regions B, D, and F and dirt must be added to regions A, C, and E. Dirt can be moved from region to region, or removed from the area entirely. Dirt can also be brought in from outside the area, if necessary. The dirt requirements are given in the tables below.

<table>
<thead>
<tr>
<th>Region</th>
<th>Dirt Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>20 tons</td>
</tr>
<tr>
<td>C</td>
<td>80 tons</td>
</tr>
<tr>
<td>E</td>
<td>40 tons</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Region</th>
<th>Extra Dirt</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>15 tons</td>
</tr>
<tr>
<td>D</td>
<td>10 tons</td>
</tr>
<tr>
<td>F</td>
<td>90 tons</td>
</tr>
</tbody>
</table>
Let $c_{ij}$ be the cost of shipping each ton of dirt from region $i$ to region $j$. Assume that the costs satisfy the triangle inequality; that is, for any three regions $i$, $j$, and $k$ the following inequality holds:

$$c_{ij} + c_{jk} \geq c_{ik}.$$  

(In other words, the shortest distance between two points is a straight line – you can’t get a cheaper cost between $i$ and $k$ by stopping somewhere in between on the way.)

Draw this scenario as a network problem. If it is a special type of network problem, specify what type it is.

3. Two nights ago, an astronomer took a picture of a small portion of the sky containing 10,000 stars. Last night, the astronomer took a picture of the same portion of sky (containing the same 10,000 stars). The stars have all moved, and by measuring the distance they have moved the astronomer can estimate their speed.

Unfortunately, it’s not quite that easy. Because the astronomer does not know the speed (or direction of motion) of the stars beforehand, all she sees are two pictures with 10,000 dots on them. There is no way for her to be sure which dot in the second picture is the same star as a certain dot in the first picture!

Although the astronomer can’t be sure which dot is which, she can make some educated guesses. Based on the size and brightness of each dot, she can estimate the probability $p_{ij}$ that star $i$ in the first picture is the same as star $j$ in the second picture. Using these probabilities, she would like to make her best guess as to which star is which.

Draw this scenario as a network problem. If it is a special type of network problem, specify what type it is.

4. Mapquest.com is an internet site that gives driving directions between any two points in the United States. Based on an internal street map of the United States, it solves a network optimization problem to find the fastest route.

(a) Describe the network problem that Mapquest.com solves (i.e., what are the nodes, what are the arcs, what are the costs and capacities on each arc, what are the supplies and demands on each node). If it is a special type of network problem, specify what type it is.

(b) Suppose you wanted to find the route with the shortest driving distance, even if it took more time to drive. How would your answer to (a) change?

(c) Suppose you wanted to find the route with the fewest number of turns (or highway changes). How would your answer to (a) change?

5. Currency traders attempt to make money by exchanging one country’s currency for another. For example, on Monday, 1 U.S. dollar could be exchanged for 1.49451
Canadian dollars. If tomorrow the exchanged rate dropped to 1.4 Canadian dollars per U.S. dollar, he could then trade his 1.49451 Canadian dollars for 1.49451/1.4 = 1.0675 U.S. dollars, a profit of 6.75%.

If a trader exchanged 1 U.S. dollar for 1.49451 Canadian dollars today and then instantly traded them back, it looks like he would get 1.49451/1.49451 = 1 U.S. dollar back. Unfortunately, this isn’t quite true, because there are fees for every transaction (purchase or sale) of currency. Instead, to reflect these fees, the exchange rate might be a little smaller in one direction (say, 1.49 Canadian dollars for a U.S. dollar) and a little larger in the other direction (say, 1/1.50 U.S. dollars per Canadian dollar). So after the two trades, the trader would have 1.49/1.50 = 0.993333 dollars at the end.

In reality, the exchange rates change constantly, not just daily. Every transaction of currency alters the exchange rate, and sometimes it takes one of the rates a short amount of time to “catch up” to the others. Therefore, for a very short time (usually less than one second) it might be possible to trade, say, 1 U.S. dollar for 1.50 Canadian dollars and then trade each Canadian dollar for 1/1.49 U.S. dollars. Just by trading back and forth, you could start with 1 dollar and end up with 1.50/1.49 = 1.00671 dollars. Thanks to the speed of electronic transactions, a fast automated computer might be able to make this exchange, say, 1,000,000 times in the one second to give you a profit of $6710 – not bad for one second of work! And if you did the trade with $1000 instead of $1, you could make $6,710,000 in that one second. This type of money-making scheme is called arbitrage.

Usually, these “obvious” opportunities for arbitrage, in which you trade one currency for another and then trade right back, are watched for and disallowed by banks. However, there are dozens of world currencies, and it might also be possible to make a more roundabout money-making trade. For example, you might be able to achieve arbitrage by trading U.S. dollars for Canadian dollars, then trading those Canadian dollars for European euros, then trading those European euros for Japanese yen, and finally trading those Japanese yen for U.S. dollars, and end up with more U.S. dollars than you started with.

Suppose you wanted to see whether there was an arbitrage opportunity using U.S dollars, Canadian dollars, euros, and yen. It is possible to formulate this problem as a network problem.

(a) Draw the appropriate network.
(b) If this is a special type of network problem, specify what type it is.

6. Library shelving (p. 400, #8).
(a) Draw the appropriate network.
(b) Solve the problem using LINDO or CPLEX. Hand in your input file along with the optimal solution.
(c) Solve the problem using LINGO or AMPL. Hand in your input file.
(d) Solve the problem using the primal simplex algorithm specialized to network problems (as we saw in class). Use $x_{04}$, $x_{08}$, and $x_{12}$ as your starting basis.