1.(a)

Variables: $x_j = \text{hours per day that machine } j \text{ runs}$

Minimize $4000 \ x_1 + 1000 \ x_2$

Subject to

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$300 \ x_1 + 100 \ x_2 \geq 3000$</td>
<td>(chemical A minimum)</td>
</tr>
<tr>
<td>$100 \ x_1 + 100 \ x_2 \geq 500$</td>
<td>(chemical B minimum)</td>
</tr>
<tr>
<td>$100 \ x_1 + 200 \ x_2 \geq 2000$</td>
<td>(chemical C minimum)</td>
</tr>
<tr>
<td>$x_1 \leq 24$</td>
<td>(at most 24 hours in a day)</td>
</tr>
<tr>
<td>$x_2 \leq 24$</td>
<td>(at most 24 hours in a day)</td>
</tr>
<tr>
<td>$x_1, x_2 \geq 0$</td>
<td>(nonnegativity)</td>
</tr>
</tbody>
</table>

(b)

Before solving the problem, we have to put it into standard form. That means we need to add one slack variable for each $\leq$ constraint, and one excess (surplus) variable for each $\geq$ constraint.

Minimize $4000 \ x_1 + 1000 \ x_2$
Subject to

\[\begin{align*}
300 \, x_1 + 100 \, x_2 - \, e_1 &= 3000 \\
100 \, x_1 + 100 \, x_2 - \, e_2 &= 500 \\
100 \, x_1 + 200 \, x_2 - \, e_3 &= 2000 \\
x_1 + s_4 &= 24 \\
x_2 + s_5 &= 24 \\
x_1, \, x_2, \, e_1, \, e_2, \, e_3, \, s_4, \, s_5 &\geq 0
\end{align*}\]

\[b\] will be the same at every iteration:

\[
\begin{bmatrix}
3000 \\
500
\end{bmatrix}
\]

\[b = \begin{bmatrix}
2000 \\
24 \\
24
\end{bmatrix}\]

Since we’re starting with the basis \{\(e_1, \, e_2, \, e_3, \, x_1, \, x_3\)\},

\[
\begin{bmatrix}
\begin{bmatrix} e_1 \end{bmatrix} & \begin{bmatrix} e_2 \end{bmatrix} & \begin{bmatrix} e_3 \end{bmatrix}
\end{bmatrix}, \quad
\begin{bmatrix}
\begin{bmatrix} x_1 \end{bmatrix} & \begin{bmatrix} x_2 \end{bmatrix}
\end{bmatrix}, \quad
\begin{bmatrix}
\begin{bmatrix} \begin{bmatrix} c_B \end{bmatrix} & \begin{bmatrix} b \end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix} -1 \, 0 \, 0 \, 300 \, 100 \\
0 \, 1 \, 0 \, 100 \, 100 \\
0 \, 0 \, -1 \, 100 \, 200 \\
0 \, 0 \, 0 \, 1 \, 0 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
\end{bmatrix}
\end{bmatrix}
\]

\[\text{ITERATION 1}\]

First, we need to find \(B^{-1}\):

\[
\begin{bmatrix}
1 \, 0 \, 0 \, -300 \, -100 \\
0 \, 1 \, 0 \, -100 \, -100 \\
0 \, 0 \, 1 \, -100 \, -200 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
1 \, 0 \, 0 \, 0 \, 1 \\
0 \, 1 \, 0 \, 0 \, 1 \\
0 \, 0 \, 1 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
0 \, 0 \, 0 \, 0 \, 1 \\
\end{bmatrix}
\]
So,

\[
\begin{bmatrix}
-1 & 0 & 0 & 300 & 100 \\
0 & -1 & 0 & 100 & 100 \\
0 & 0 & -1 & 100 & 200 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[B^{-1} = \begin{bmatrix}
0 & 0 & 0 & 4000 & 1000 \\
0 & 0 & -1 & 100 & 200 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Reduced costs for nonbasic variables:

It will be easier to calculate \(c_B B^{-1}\) once, instead of repeating the calculation for every nonbasic variable.

\[c_B B^{-1} = \begin{bmatrix}
-1 & 0 & 0 & 300 & 100 \\
0 & -1 & 0 & 100 & 100 \\
0 & 0 & -1 & 100 & 200 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
s_4: \ c_{s4} - c_B B^{-1} a_{s4} = 0 - 4000 = -4000
\]

\[
s_5: \ c_{s5} - c_B B^{-1} a_{s5} = 0 - 1000 = -1000
\]

Since we’re minimizing, variables with negative reduced cost will help the objective. So, we can choose either one to be our entering variable. Let’s choose \(s_4\).

To find the variable that leaves the basis, we need to find the current basic variable that reaches zero first.

\[x_B = B^{-1} b = \begin{bmatrix}
-1 & 0 & 0 & 300 & 100 \\
0 & -1 & 0 & 100 & 100 \\
0 & 0 & -1 & 100 & 200 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
3000 \\
500 \\
2000 \\
24 \\
24 \\
\end{bmatrix} = \begin{bmatrix}
6600 \\
4300 \\
5200 \\
24 \\
24 \\
\end{bmatrix}
\]

We also need to find the rate of change of each basic variable if \(s_4\) enters.

\[rate = B^{-1} a_{s4} = -\begin{bmatrix}
-1 & 0 & 0 & 300 & 100 \\
0 & -1 & 0 & 100 & 100 \\
0 & 0 & -1 & 100 & 200 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
\end{bmatrix} = -\begin{bmatrix}
-300 \\
-100 \\
-100 \\
-1 \\
0 \\
\end{bmatrix}
\]

So, how far can \(s_4\) enter?

Limit from \(e_1 = 6600/300 = 22\)
Limit from \(e_2 = 4300/100 = 43\)
Limit from $e_3 = \frac{5200}{100} = 52$
Limit from $x_1 = \frac{24}{1} = 24$
Limit from $x_2 = \frac{24}{0} = \text{infinite}$

So the smallest limit is from $e_1$. Therefore, when $s_4$ enters the basis, $e_1$ leaves.

ITERATION 2

Now, our new set of basic variables is $\{s_4, e_2, e_3, x_1, x_2\}$.

The new $B^{-1}$ is just $E$ times the old $B^{-1}$, where $E$ is the identity matrix, with the first column (because the first variable left the basis) replaced by a special column based on variable $s_4$ (because $s_4$ entered the basis).

From last iteration, we know that

$$B^{-1}a_{s4} = \begin{bmatrix} 300 \\ 100 \\ 1 \\ 0 \end{bmatrix}$$

So to get the special column, we divide each entry but the first by $-300$ (because the leaving variable was the first one) and the first entry is just $1/300$.

$$E = \begin{bmatrix} \frac{1}{300} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 & 0 \\ -\frac{1}{300} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The new $B^{-1}$ is just $E$ times the old $B^{-1}$:

$$B^{-1} = \begin{bmatrix} \frac{1}{300} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 & 0 \\ -\frac{1}{300} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 300 & 100 \\ -1 & 0 & 0 & 100 & 100 \\ 0 & -1 & 100 & 200 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E \begin{bmatrix} \frac{1}{300} & 0 & 0 & 1 & \frac{1}{3} \\ \frac{1}{3} & -1 & 0 & 0 & 66^{2/3} \\ \frac{1}{3} & 0 & -1 & 0 & 166^{2/3} \\ \frac{1}{300} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(Whew! Wasn’t that much easier than inverting $B$ again from scratch?)
Reduced costs for nonbasic variables:

It will be easier to calculate $c_B B^{-1}$ once, instead of repeating the calculation for every nonbasic variable.

$$
c_{B^{-1}} = \begin{bmatrix}
\frac{-1}{300} & 0 & 0 & 1 & \frac{1}{3} \\
\frac{1}{3} & -1 & 0 & 0 & 66\frac{2}{3} \\
\frac{1}{300} & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
4000 \\
1000 \\
\frac{1}{3} \\
0 \\
\frac{1}{300}
\end{bmatrix}
= \begin{bmatrix}
13\frac{1}{3} & 0 & 0 & -333\frac{1}{3}
\end{bmatrix}
$$

\(e_1: c_{e_1} - c_{B^{-1}}a_{e_1} = 0 - -13\frac{1}{3} = 13\frac{1}{3}\)

\(s_5: c_{s_5} - c_{B^{-1}}a_{s_5} = 0 - -333\frac{1}{3} = 333\frac{1}{3}\)

There are no variables with negative reduced costs. Therefore, this is the optimal basis. The optimal solution $x_B = B^{-1}b$.

$$
x_B = B^{-1}b = \begin{bmatrix}
\frac{-1}{300} & 0 & 0 & 1 & \frac{1}{3} \\
\frac{1}{3} & -1 & 0 & 0 & 66\frac{2}{3} \\
\frac{1}{300} & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3000 \\
500 \\
24 \\
24
\end{bmatrix}
= \begin{bmatrix}
22 \\
2100 \\
24 \\
24
\end{bmatrix}
$$

So, $s_4 = 22$, $e_2 = 2100$, $e_3 = 3000$, $x_1 = 2$, and $x_2 = 24$. Our two nonbasic variables, $e_1$ and $s_5$, are both zero.

The objective value is $c_B x_B = $8000 + $24000 = $32000.

\[(d)\]

\[(i)\]

If the engineer can get us 200 extra gallons of A per day, it’s like we’re getting that much extra, so we would only need to run the machines long enough to make $3000 - 200 = 2800$ gallons.

Remember that $c_B B^{-1}$ is the vector of shadow prices. The shadow price of chemical A is $13\frac{1}{3}$ (from above), so the objective would change by $(13\frac{1}{3})(-200) = -2666\frac{2}{3}$.

That’s on a per-day basis, so over 365 days the company would save $(365)(2666\frac{2}{3}) = $973,333.33 – a very nice return on the engineer’s $50,000 salary! We get our money back in $50000/2666\frac{2}{3} = less than 14 days.

However, we need to be sure that the shadow price of $13\frac{1}{3}$ is valid over the whole range of a 200-unit decrease. Recall that the shadow price is only valid until a new
strategy becomes optimal; in other words, when one of the basic variables becomes zero.

If we replace 3000 with 3000 + $\delta$ in the b-vector, the equation $x_B = B^{-1}b$ from above gives us

\[
\begin{bmatrix}
22 - \frac{\delta}{300} \\
2100 + \frac{\delta}{3}
\end{bmatrix}
\begin{bmatrix}
3000 + \frac{\delta}{3} \\
2 + \frac{\delta}{300} \\
24
\end{bmatrix}
\]

Setting $x_B \geq 0$ gives the following four inequalities:

\[\delta \leq 6600 \quad \text{(from the first row)}\]
\[\delta \geq -6300 \quad \text{(second row)}\]
\[\delta \geq -9000 \quad \text{(third row)}\]
\[\delta \geq -600 \quad \text{(fourth row)}\]

The last row contains no $\delta$ term, so it does not give us any restriction.

Therefore, as long as the right-hand side does not decrease by more than 600 or increase by more than 6600, the current strategy remains optimal (and so the shadow price remains valid). A 200-gallon decrease is within this range, so our answer of $973,333.33$ is correct.

(ii)

In this case, we’re changing the right-hand side of the machine 2 hours constraint. Instead of having a 24-hour limit, we’ll now have a 48-hour limit (2 machines, each can operate 24 hours).

If we replace 24 with $24 + \delta$ in the b-vector, the equation $x_B = B^{-1}b$ from above gives us

\[
\begin{bmatrix}
22 + \frac{\delta}{3} \\
2100 + \frac{2000\delta}{3} \\
3000 + \frac{5000\delta}{3} \\
2 - \frac{\delta}{3} \\
24 + \delta
\end{bmatrix}
\]

So, when we set $x_B \geq 0$, we get the following 5 inequalities:

\[\delta \geq -66\]
\[\delta \geq -31^{1/2}\]
\( \delta \geq -18 \)
\( \delta \leq 6 \)
\( \delta \geq -24 \)

Therefore, as long as \(-18 \leq \delta \leq 6\), the shadow price will remain valid. Unfortunately, our value of \(\delta\) is 24 (we are adding 24 hours of available machine time), so the shadow price will not be valid. To find the total effect, we need to re-optimize.

Luckily, re-optimization isn’t so difficult. Let’s start from the same basis we began our initial optimization from. We know that \(B\) hasn’t changed (only \(b\) has), so we don’t have to re-invert the matrix; we already know \(B^{-1}\).

\[
B^{-1} = \begin{bmatrix}
-1 & 0 & 0 & 300 & 100 \\
0 & -1 & 0 & 100 & 100 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

The reduced costs for nonbasic variables are also the same, since they don’t depend on \(b\).

\(s_4: \quad c_s - c_BB^{-1}a_s = 0 - 4000 = -4000 \) (known from part (c))
\(s_5: \quad c_s - c_BB^{-1}a_s = 0 - 1000 = -1000 \) (known from part (c))

Again, let’s choose \(s_4\) as our entering variable.

Here’s the first time we need to do some new computations. To find the variable that leaves the basis, we need to find the current basic variable that reaches zero first, and that depends on \(b\).

\[
x_B = B^{-1}b = \begin{bmatrix}
-1 & 0 & 0 & 300 & 100 \\
0 & -1 & 0 & 100 & 100 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
3000 \\
500 \\
24 \\
48 \\
\end{bmatrix}
= \begin{bmatrix}
9000 \\
6700 \\
24 \\
48 \\
\end{bmatrix}
\] (new computation)

We also need to find the rate of change of each basic variable if \(s_5\) enters, but that’s the same as before.

\[
rate = B^{-1}a_s = -\begin{bmatrix}
-1 & 0 & 0 & 300 & 100 \\
0 & -1 & 0 & 100 & 100 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
-300 \\
-100 \\
-1 \\
0 \\
\end{bmatrix}
\] (known from part (c))

So, how far can \(s_4\) enter?

Limit from \(e_1 = 9000/300 = 30\)
Limit from $e_2 = 6700/100 = 67$
Limit from $e_3 = 10000/100 = 100$
Limit from $x_1 = 24/1 = 24$
Limit from $x_2 = 48/0 = \text{infinity}$

So the smallest limit is from still $x_1$. Therefore, when $s_4$ enters the basis, $x_1$ leaves.

ITERATION 2

Now, our new set of basic variables is \{e_1, e_2, e_3, s_4, x_2\}.

The new $B^{-1}$ is just $E$ times the old $B^{-1}$, where $E$ is the identity matrix, with the fourth column (because the fourth variable left the basis) replaced by a special column based on variable $s_4$ (because $s_4$ entered the basis).

From before, we know that

$$
\begin{bmatrix}
300 \\
100 \\
100 \\
1 \\
0
\end{bmatrix}
= \\
\begin{bmatrix}
100 \\
1 \\
0
\end{bmatrix}
$$
(known from part (c))

So to get the special column, we divide each entry but the fourth by -1 (because the leaving variable was the fourth one) and the fourth entry is just 1/1.

$$
\begin{bmatrix}
1 & 0 & 0 & -300 & 0 \\
0 & 1 & 0 & -100 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
= \\
\begin{bmatrix}
1 & 0 & 0 & -300 & 0 \\
0 & 1 & 0 & -100 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$
(new computation)

The new $B^{-1}$ is just $E$ times the old $B^{-1}$.

$$
\begin{bmatrix}
1 & 0 & 0 & -300 & 0 \\
0 & 1 & 0 & -100 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
= \\
\begin{bmatrix}
1 & 0 & 0 & 300 & 100 \\
0 & 1 & 0 & 100 & 100 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$
(new)

Reduced costs for nonbasic variables:

It will be easier to calculate $c_B B^{-1}$ once, instead of repeating the calculation for every nonbasic variable.

$$
\begin{bmatrix}
-1 & 0 & 0 & 0 & 100
\end{bmatrix}
$$
\[
\begin{bmatrix}
0 & -1 & 0 & 0 & 100 \\
0 & 0 & -1 & 0 & 200 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
c_B B^{-1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1000 \\
0 & 0 & -1 & 0 & 2000 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1000 \\
0 & 0 & 0 & 0 & 1000
\end{bmatrix}
\]

\[
x_1 : \quad c_{x1} - c_B B^{-1} a_{x1} = 4000 - 0 = 4000
\]
\[
s_5 : \quad c_{s5} - c_B B^{-1} a_{s5} = 0 - 1000 = -1000
\]

\(s_5\) is the only variable with a negative reduced cost, so it will enter the basis. Let’s see how far we can go entering \(s_5\).

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 & 100 \\
0 & -1 & 0 & 0 & 100 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3000 \\
500 \\
24 \\
48
\end{bmatrix}
= \begin{bmatrix}
1800 \\
4300 \\
24 \\
48
\end{bmatrix}
\]

\[
x_B = B^{-1} b = \begin{bmatrix}
0 & 0 & -1 & 0 & 200 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2000 \\
24 \\
48
\end{bmatrix}
= \begin{bmatrix}
7600 \\
24 \\
48
\end{bmatrix}
\]

\[
rate = B^{-1} a_{s5} = -\begin{bmatrix}
0 & 0 & -1 & 0 & 200 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix}
\]

Limit from \(e_1\): \(1800/100 = 18\)
Limit from \(e_2\): \(4300/100 = 43\)
Limit from \(e_3\): \(7600/200 = 38\)
Limit from \(s_4\): \(24/0 = \text{infinity}\)
Limit from \(x_2\): \(48/1 = 48\)

\(e_1\) has the smallest limit, so \(e_1\) is the variable that will leave the basis when \(s_5\) enters.

**ITERATION 3**

Now, our new set of basic variables is \{\(s_5, e_2, e_3, s_4, x_2\}\).

The new \(B^{-1}\) is just \(E\) times the old \(B^{-1}\), where \(E\) is the identity matrix, with the first column (because the first variable left the basis) replaced by a special column based on variable \(s_5\) (because \(s_5\) entered the basis).

From before, we know that

\[
\begin{bmatrix}
100 \\
100
\end{bmatrix}
\]

\[
B^{-1} a_{s5} = \begin{bmatrix}
200 \\
0 \\
1
\end{bmatrix}
\]
So to get the special column, we divide each entry but the first by -100 (because the leaving variable was the first one) and the first entry is just 1/100.

\[
\begin{bmatrix}
\frac{1}{100} & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\frac{-1}{100} & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[E = \begin{bmatrix}
\frac{1}{100} & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\frac{-1}{100} & 0 & 0 & 0 & 1 \\
\end{bmatrix}\]

The new \(B^{-1}\) is just \(E\) times the old \(B^{-1}\).

\[
\begin{bmatrix}
\frac{1}{100} & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\frac{-1}{100} & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\frac{1}{100} & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 \\
2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\frac{1}{100} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
\frac{-1}{100} & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 \\
2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\frac{1}{100} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Reduced costs for nonbasic variables:

It will be easier to calculate \(c_{b}B^{-1}\) once, instead of repeating the calculation for every nonbasic variable.

\[
\begin{bmatrix}
\frac{-1}{100} & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 \\
2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\frac{1}{100} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 1000 \\
2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & 0 \\
\frac{1}{100} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
10 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[x_1 : \ c_{x1} - c_{b}B^{-1}a_{x1} = 4000 - 3000 = 1000\]
\[e_1 : \ c_{e1} - c_{b}B^{-1}a_{e1} = 0 - -10 = 10\]

There are no variables with negative reduced cost, so this is the optimal solution. The values of the nonbasic variables \(x_1\) and \(e_1\) are zero. The values of the basic variables are \(x_B = B^{-1}b\).

\[
\begin{bmatrix}
\frac{-1}{100} & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 \\
2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\frac{1}{100} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
3000 \\
500 \\
2000 \\
24 \\
48 \\
\end{bmatrix}
= \begin{bmatrix}
18 \\
2500 \\
4000 \\
24 \\
30 \\
\end{bmatrix}
\]

So, \(s_5 = 18, e_2 = 2500, e_3 = 4000, s_4 = 24, x_2 = 30\).
Now… I know it has been a long process, but we’re trying to figure out how much better the objective value is now that we’ve doubled the available hours on machine 2.

The new objective value is \( c_B x_B = $30000 \). The old objective was $32000, so the extra machine saves us $2000 per day. On an annual basis, that’s $2000 \times 365 = $730,000 – another good return on our annual $50,000 investment. We get our money back in just 25 days.

(iii)

Adding a new type of machine would give us another production option – in essence, another variable, \( x_3 \). So, we can start at the optimal vasis of part (c), and check the reduced cost of this new variable.

Reduced cost of \( x_3 \): \( c_{x_3} - c_B B^{-1} a_{x_3} \).

From part (c), we know that \( c_B B^{-1} = \begin{bmatrix} 13^{1/3} & 0 & 0 & -333^{1/3} \end{bmatrix} \). The cost of each hour of operation for machine 3 is $3000, so \( c_{x_3} = 3000 \). \( a_{x_3} \) is the coefficient of \( x_3 \) in each constraint if we were to add it to the formulation. Since the machine makes 200 gallons of A, 300 gallons of B, and 300 gallons of C per hour,

\[
\begin{bmatrix}
200 \\
300 \\
0 \\
0 
\end{bmatrix}
\]

\( a_3 = \begin{bmatrix} 200 \\ 300 \\ 0 \\ 0 \end{bmatrix} \)

So, \( c_{x_3} - c_B B^{-1} a_{x_3} = 3000 - 2666^{2/3} = 333^{1/3} \). Since the reduced cost is positive, we would not enter \( x_3 \) into the basis even if it was available, so paying for it makes no sense. We would never use it, so it would never make us any profit and we would never recoup our $50,000 annual investment.

(iv)

Since (i) gives the best return on investment, (i) would probably be the best use of it right now.

(v)

If the reduced cost ever became negative, we would enter it into the basis. Since the current reduced cost is \( 333^{1/3} \), the operating cost of machine 3 would have to be reduced by \( 333^{1/3} \). \( 3000 - 333^{1/3} = 2666^{2/3} \), or $2666.67 per day.

2.(a)
Variables:  
\(x_j\) = gallons of fuel at takeoff from city \(j\)  
\(y_j\) = gallons of fuel when landing at city \(j\)  
\(z_j\) = gallons of fuel purchased at city \(j\)

Minimize  
\[88 \times L + 15 \times H + 105 \times N + 95 \times M\]

Subject to
\[y_H = x_L - 1500(1 + (x_L + y_H)/2000)\]  
\[y_N = x_H - 1700(1 + (x_H + y_N)/2000)\]  
\[y_M = x_N - 1300(1 + (x_N + y_M)/2000)\]  
\[y_L = x_M - 2700(1 + (x_M + y_L)/2000)\]  
(all of these constraints say the amount of fuel left upon landing equals the starting amount minus what’s used)

\[x_H = y_H + z_H\]  
\[x_N = y_N + z_N\]  
\[x_M = y_M + z_M\]  
\[x_L = y_L + z_L\]  
(these constraints all say that the amount of fuel the plane takes off with equals whatever it landed with plus whatever is bought)

\[y_H \geq 600\]  
\[y_N \geq 600\]  
\[y_M \geq 600\]  
\[y_L \geq 600\]  
(say that upon landing the plane must have at least 600 gallons of fuel)

\[z_H \leq 10000\]  
\[z_N \leq 10000\]  
\[z_M \leq 10000\]  
\[z_L \leq 10000\]  
(say that no more than 10,000 gallons of fuel can be purchased an once)

\[x_H \leq 12000\]  
\[x_N \leq 12000\]  
\[x_M \leq 12000\]  
\[x_L \leq 12000\]  
(say that the plane can never hold more than 12,000 gallons of fuel)

all \(x_j, y_j, z_j \geq 0\)  
(nonnegativity)

This formulation has 20 constraints and 12 variables. In standard form, each of the 12 inequalities would have an excess or slack variable, for a total of 24 variables.

(b)

Since the matrix \(B\) has one row per constraint and one column for every basic variable (and there’s one basic variable per constraint), it would be a 20x20 matrix. Trust me, inverting a 20x20 matrix by hand is not fun!

(c)

We can solve the first set of constraints for the \(y\)-variables in terms of the \(x\)-variables:
In general,
\[ y_j = x_i - d(1 + (x_i + y_j)/2000) , \]
\[ = (4000-d)/(4000+d) x_i - (4000*d)/(4000+d) \]

We also need to substitute this expression into the \( y_j \geq 0 \) constraints, so it might seem like we’re not eliminating any constraints. However, because we already have constraints saying \( y_j \geq 600 \), we can remove the \( y_j \geq 0 \) constraints without losing anything.

Now, use the second set of constraints to solve for the x-variables in terms of the z-variables:
\[ x_j = y_j + z_j = (4000-d)/(4000+d) x_i - (4000*d)/(4000+d) + z_j \]

We can now back-substitute into the last equation, so that both x and y are in terms of z. We can make the same substitution into the third and fourth set of constraints, and the objective.

Again, we would usually have to substitute in for the \( x_j \geq 0 \) constraints. However, since we know \( y_j \geq 600 \) and \( z_j \geq 0 \) are still part of our constraint set, it must be true that their sum will automatically be \( \geq 0 \), so we can again eliminate the \( x_j \geq 0 \) constraints.

Now, our formulation consists only of the last three sets of constraints, all in terms of the z-variables. So, we have 12 constraints and 4 variables. If we put the problem in standard form, we’ll add one slack or excess variable for each constraint, for a total of 16 variables.

Now, we only need to invert a 12x12 matrix – it’s still not much fun, but it’s way better than 20x20!

[NOTE: It’s also faster and easier for computers to work with smaller matrices. So, the best linear programming solution software packages will automatically substitute out all the equality constraints for you.]

(d)

I don’t know about you, but I say NEITHER! 12x12 is still too big to want to solve by hand. That’s why we’ll soon be learning how to use linear programming software to create models and solve problems.