Using Shapley Value To Allocate Savings in a Supply Chain

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Abstract

Consider two retailers, whose inventory is provided by a common supplier who bears all the inventory risk. We model the relationship among the retailers and supplier as a single-period cooperative game in which the players can form inventory-pooling coalitions. Using the Shapley value to allocate the profit, we analyze various schemes by which the supplier might pool inventory she holds for the retailers. We find, among other things, that the Shapley value allocations are individually rational and are guaranteed to coordinate the supply chain; but they may be perceived as unfair in that the retailers' allocations can, in some situations, exceed their contribution to supply chain profit. Finally we analyze the effects of demand variance and asymmetric service level requirements on the allocations.

1 An Inventory Centralization Model

Consider an electronics manufacturing services provider (EMS), who keeps inventory of cpu chips for two or more competing original equipment manufacturers (OEM). The current inventory policy dictated by the OEMs is to keep each company's inventory physically separated. Is this the most profitable inventory policy for the EMS? Furthermore, is the most profitable inventory policy for the EMS also the most profitable for her customers?

In general we are interested in knowing whether a supplier should pool inventory held for her customers (the retailers). If so, what will be the benefits and how should they be shared over the supply chain? Will a customer (retailer) who requires a higher level of service be indirectly subsidizing a competitor who would accept a lower level of service? We explore such questions in the following 2-echelon supply chain using a single-period model.

Consider two retailers selling a single product procured form a single, common supplier. Even though there may be more suppliers providing the same product in the larger supply chain, we consider a situation where the retailers already chose to work with a particular supplier. For example companies in the electronics industry prefer to have a sole supplier for each product whenever possible [6]. The retailers face uncertain demand and do not carry inventory. When they observe demand, they place an order at the supplier and receive shipments without significant delay. Ownership passes from the supplier to a retailer after the retailer places the order and pays for the product and so the supplier bears all the inventory risk. Sales are lost to the retailers in case of a stock-out at the supplier. (There is no backlogging.) To service the retailers, the supplier either keeps inventory reserved for each of her customers or else pools inventory to share among all of her customers.

Inventory-pooling is known to reduce costs and so increases profits for the supply chain party that owns the inventory, in this case, the supplier [12]. However, the retailers may object to inventory-pooling because of two concerns. First is the concern of how inventory will be allocated among the retailers when there are shortages. With reserved inventory, the retailer can control his risk of stock-out by specifying minimum-inventory levels to be held by the supplier. But if the retailers draw on a common, pooled inventory, which of the competing retailers has priority when requesting the last of the inventory? Any inventory-pooling contract will need to address this issue either directly (by specifying a stock-rationing mechanism) or indirectly (by specifying reservation profits to the parties such that their profits are at least as much as their before-pooling profits).

The second concern is how much information should be shared in the supply chain to facilitate inventory-pooling. In the case of reserved inventories, each company shares demand information only with the supplier. However, in the case of inventory pooling, a company can, by observing his own service level, infer something about the demand faced by the competitor with whom he is sharing inventory.

In this paper, we first consider supply chain members with varying degrees of power, where we take power to be the ability to dictate a strategy of pooling or no pooling. We show that the supply-chain-optimal inventory level cannot be attained under powerful retailers who preclude pooling or a powerful supplier who pools inventory to maximize her profits. Furthermore, retailers may lose profits (compared to the case without inventory pooling) when the supplier pools inventory subject to the retailers' service constraints. We conclude that the frequently used service measure, probability of no stock-out, does not induce supply-chain-optimal inventory levels in the system.

Instead we propose a *value-sharing method* based on Shapley value from cooperative game theory and derive closed-form expressions of the Shapley values. We find that the Shapley value induces coordination and the allocations under this mechanism satisfy individual rationality conditions for all players and belong to the core of the game. Though stable, an allocation based on Shapley value may induce envy among some players. In particular, we find that the allocation mechanism may be interpreted as "unfair" by some players. We show that the mechanism favors retailers in the sense that retailer allocations may exceed their contribution to total supply chain profit at the expense of the supplier.

Under the proposed contract, the retailers prefer to form pooling coalitions with retailers with either very high or very low service requirements. Up to a threshold service level a retailer prefers to be the one requesting the higher service level because it ensures him the greater share of total profits. Beyond the threshold level a coalition partner with very high service requirements forces the supplier to overstock, increasing sales for both of the retailers. We also show that when the supplier has the power to maximize her profits by manipulating the service levels she provides for the retailers, the retailer with lower demand variance has a better chance of increasing his profits. The Shapley value scheme rewards the retailer introducing less risk into the supply chain and one can reasonably argue that this is "fair".

In the next section, we survey related literature and position our model. In Section 3 we analyze the supply chain profit and its distribution among parties of varying degrees of power. We then introduce the Shapley value profit allocation mechanism in Section 4 and explore the Shapley value allocations and their properties in Section 5. In Section 6, we discuss the possible instabilities that may be caused by the Shapley value allocation scheme. Finally, in Section 7, we analyze the question "With whom to form a coalition" from the (different) perspectives of a retailer and the supplier given the service level constraints of each of the retailers. We conclude with a discussion of our findings and future research directions.

2 Literature Review

Most of the cost models analyzed up to now are extensions of the classical news vendor problem, for which Porteus [24] provides a review. The literature on inventory pooling (also known as risk pooling) can be classified under three headings.

- Component commonality
- Inventory rationing/transsphipment in single echelon supply chains
- Inventory and risk pooling in multi-echelon supply chains

Component Commonality

If end products share common components, safety stock can be reduced and service levels maintained by pooling inventory of common parts. The work-to-date on component commonality concentrates merely on changes in safety stock levels and does not consider the benefits of pooling to different members of the supply chain nor how they should be shared. Baker, Magazine, and Nuttle [5] consider a two product system with service level constraints and where the objective is to minimize total safety stock. They show that total safety stock (common and specialized) drops after pooling; however total stock of specialized parts increases. Gerchak, Magazine, and Gamble [15] extend these results to a profit maximization setting. Finally, Gerchak and Henig [14] extend these models to a multi-period setting and show that myopic policies are optimal for the infinite horizon models.

Inventory Rationing in Single Echelon Supply Chains

Inventory rationing defines the rules of how to allocate total inventory to n different members of the same echelon of a supply chain in case of a shortage (shortage for all members or shortage for some and overage for others). This can either be done through transshipments among supply chain members carrying decentralized inventory or by defining rules to allocate inventory when it is centralized at a single location. This approach is different from our work in that it concentrates on one of the echelons only.

One question regarding centralized inventories that has received attention in the literature is whether total inventory level in the supply chain decreases after pooling. Gerchak and Mossman [16], Pasternack and Drezner [22], and Yang and Schrage [32] show that, contrary to intuition, this is not always the case. These papers present inventory increase as an undesired outcome of pooling. We show that increasing inventory may be beneficial for the supply chain as a whole because it also increases sales. In addition, we show that if the service level constraints are binding, inventory will not increase due to pooling. Conversely, Tagaras [31] looks at a two retailer model and shows that if the total reserved safety stock for the two retailers is pooled and used to replenish both of the retailers from a central location, service levels at both of the retailers will increase.

One stream of papers analyzes the inventory-sharing problem as a transshipment problem among different players at the same echelon, possibly with positive transshipment costs. These papers are more closely related to our work in that they consider decentralized systems, but they differ from our work in that they concentrate on different players within the same echelon. Anupindi, Bassok, and Zemel analyze the problem in a cooperative game theoretic framework [2]. They propose a modified duality-based allocation mechanism that achieves the profit level of the centralized system. Granot and Sosic [17] extend their work by relaxing an assumption on the amount of residual inventory available for transshipments among the retailers. Rudi, Kapur, and Pyke [27] analyze a similar problem with only two retailers. Instead of fixing the transshipment prices like Anupindi et al. do, they let the transshipment prices be variable and try to come up with prices that would coordinate the supply chain.

In addition to allocation of parts in case of shortages, allocation of costs to supply chain members is an important issue in centralized inventory systems. Gerchak and Gupta [13] analyze this question for a system with an EOQ-based inventory policy and argue that allocating costs with respect to volume of demand or contribution to total cost may result in unacceptable cost allocations for some parties. They propose an allocation mechanism that allocates costs based on stand-alone costs. In his note on Gerchak and Gupta's paper, Robinson [26] proposes the concept of core as a possible fair cost allocation scheme and provides a numerical example. Hartman and Dror [18, 19] also discuss core allocations and and in [18] compare several cost allocation methods (one of which is Shapley value) on a numerical example. However, [26], [18], and [19] do not analyze the operational properties of the proposed allocation mechanisms.

Inventory Pooling in Multi-Echelon Supply Chains

Of the existing literature, the work that is closest to our work is that of Anupindi and Bassok [1]. They consider a two level supply chain with a single manufacturer and two retailers. Unlike our model, the inventory decision is made by the retailers without constraining service levels and the retailers bear all the inventory risk. They model a system where only a fraction of the customers are willing to wait for a delivery from another retailer. They show that under this setting, the manufacturer may not always benefit from inventory pooling because total sales may drop. They discuss the possibility of optimizing wholesale prices or introducing holding cost subsidies as methods for coordinating the supply chain. Dong and Rudi [11] extend the model of Anupindi et al. [2] to a two echelon supply chain. Similar to our objective, they explore whether transshipments, which are beneficial for the retailers, are also beneficial for the upstream manufacturer. However, in their model the manufacturer does not hold inventory and the retailers make the transshipment decisions.

As in our work, Netessine and Rudi [21] consider a model where the supplier bears all the inventory risk. Although they also consider a two-echelon system, the second echelon consists of a single retailer. In their model, the retailer is merely an intermediary between the end customer and the supplier and functions only to expand the customer base through marketing effort. The authors conjecture that the risk-pooling effect that will be observed in the case of multiple retailers will make this kind of business model even more profitable. However, we will show that a supplier who carries out inventory pooling in order to maximize her own profit may actually reduce the total supply chain profit.

Finally, Plambeck and Taylor [23] consider capacity rather than inventory pooling. They consider a two-stage model where the first stage is a competitive game on capacity investment and the second stage is the cooperative stage where the firms pool inventory and determine the division of profit. The second stage of their model is similar to ours in that a cooperative game ensues from the capacity pooling interactions but different from ours in the profit-allocation rule used.

This paper may also be considered to lie within the literature on supply chain coordinating contracts, of which the chapter by Cachon [8] provides an excellent review (see especially the second section). A recent paper by Raghunathan [25] is relevant to this paper in terms of the methodology employed. Raghunathan also utilizes Shapley value as an allocation mechanism, but the subject of his paper is information sharing rather than inventory pooling.

3 Inventory Pooling: Definitions and Preliminary Results

Consider a supply chain with a single supplier and two retailers as in Figure 1. The retailers require a minimum service level from the supplier and the service level is defined as the probability of no stock-out. How the retailers' minimum service level requirements are set is exogenous to our model. For example the electronics-industry standard is that the supplier carries a minimum of two weeks' inventory for each customer [6]. In industries where such standards exist, the minimum service level can be defined as one corresponding to this standard. Even when the supplier and each retailer rather negotiate on the service level, we only model the interactions that take place after the service levels are decided on. The minimum service level information is shared only with the supplier and since the service levels are set exogenous to our model, we assume the retailers cannot provide false information to gain advantages.

Each retailer observes local demand, places an order with the supplier, pays a per-unit-price, and receives the inventory immediately (zero lead-time). The supplier manufactures or buys the product and holds it in inventory at her expense until an order is placed from the retailer(s). The objective of each is to maximize her single period profits. Retailer profit only depends on expected sales since the retailers do not hold inventory.

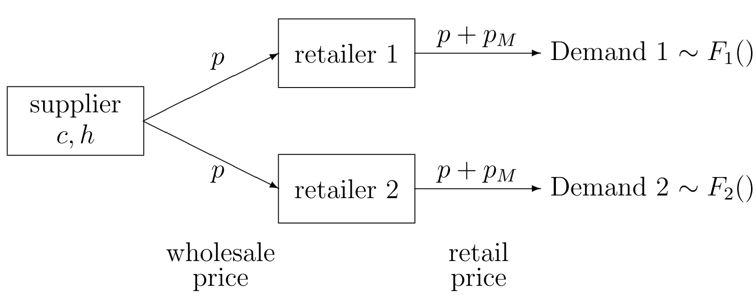

Figure 1: Sample 2-echelon supply chain and relevant cost and revenue parameters

Let p be the wholesale price the supplier charges to the retailers, c be the procurement/ manufacturing cost per unit, h be the holding cost per unit (or we can think of h as the disposal cost), and p_M be the markup on wholesale price the retailers charge. We assume that the cost and revenue parameters are common knowledge to all of the supply chain players. End customer demand is independent at the retailers and we assume the probability distributions of the demand functions are known. Let $F_i(\cdot)$ denote the cumulative demand distribution for retailer i ($i = 1, 2$). $F_i(\cdot)$ is strictly increasing and differentiable (with pdf $f_i(\cdot)$ over the interval $[0, \beta)$ where $\beta = \inf\{y : F(y) = 1\}$ (β can be ∞)).

We look at the inventory holding problem among the supplier and the two retailers in two different perspectives: the supplier holds *reserved* inventory separately for both of the players or inventory at the supplier is pooled and is *shareable* by the retailers. The total supply chain profit and its allocation among supply chain partners depend on who owns the supplier and the retailers and who makes the pooling decision. We consider the following scenarios:

- When powerful retailers forbid pooling
- When a powerful supplier pools inventory
- When a centralized supply chain makes globally optimal pooling decisions
- When a weak supplier pools inventory subject to a service contract

3.1 Powerful Retailers: No Inventory Pooling

In this scenario the retailers are powerful enough to prevent inventory pooling at the supplier. Retailers may insist on a reserved-inventory policy if the product in question is scarce (like Intel chips) and there is ambiguity about how the scarce product would be allocated or if they fear they may be underwriting the service level of a competitor. The objective of the supplier is the maximization of expected profit, which is defined as expected revenue less the expected holding (or disposal) cost and the procurement (or manufacturing) cost subject to the service level constraints. Let x_i be the stock level kept for retailer i, S_i be the expected sales at retailer i, and H_i be excess stock in retailer i's stock. For each retailer, the supplier sets inventory levels to maximize profit by solving the problem as stated in Expression 1.

$$\begin{aligned} &\max\ p\ S_i - h\ H_i - c\ x_i \\ &\text{s.t. } F_i(x_i) \geq \underline{\rho}_i \end{aligned} \tag{1}$$

where $\underline{\rho}_i =$ minimum acceptable probability of no-stockout for retailer i (or "service level").

Since the retailers do not hold inventory, their expected profit is equivalent to markup times expected sales. Each retailer's expected profit is as given in Expression 2.

$$p_M S_i \tag{2}$$

Expression 1, without the service level constraint, is the news vendor problem [30]. It is well-known that the profit-maximizing stocking level for the supplier facing demand with distribution $F(\cdot)$ is $F^{-1}(\frac{p-c}{p+h})$. The optimal stocking level corresponds to a service level of $(\frac{p-c}{p+h})$, which we call the critical ratio. The critical ratio corresponds to the probability of no stock-out, also known as Type-1 service measure. In this paper, unless otherwise specified, service level always denotes Type-1 service level.

The optimal stocking level is $F^{-1}\left(\max\left(\underline{\rho}, \frac{p-c}{p+h}\right)\right)$ when service level constraints are present and the total stock supplier must hold is $\sum_i F_i^{-1}\left(\max\left(\underline{\rho}_i, \frac{p-c}{p+h}\right)\right)$. This means that if the required service level is higher than the critical ratio then the inventory level is found such that the service constraint is binding. Service level is an increasing function of inventory and expected profit is a concave function of inventory. Therefore, whenever the required service level is higher than the critical ratio, the supplier ends up with less than optimum profit. If the service level requirements of the retailers are in the range $\left(0, \frac{p-c}{p+h}\right)$ then it is optimal for the supplier to provide higher than required service. However, beyond $\frac{p-c}{p+h}$, the supplier loses money if she provides higher service to the retailers.

Examining the structure of the optimal decision, one may observe the following:

- When the profit margin of the supplier $(p-c)$ is small or when the holding cost h is large relative to the price p, it is costlier for the supplier to provide higher-than-required service to the retailers. Therefore, utilizing the "optimum" method of pooling becomes more important.

- Like service level, expected sales is an increasing function of total stock level. Therefore, in the region, $\rho \in (0, \frac{p-c}{p+h})$, the retailers' expected sales are greater than or equal to what their service level guarantees them. Beyond $\frac{p-c}{p+h}$, however, they get exactly what they ask for because higher stock levels are not optimal for the supplier.

3.2 Inventory Pooling by a Powerful Supplier

When the supplier pools inventory to be shared by the two retailers, she effectively makes the inventory decision based on the cumulative demand $F_c(\cdot) = F_1(\cdot) * F_2(\cdot)$. Let S_c be expected cumulative sales, H_c be the expected cumulative excess stock, and x_c be the stock level. The supplier's problem is

$$\max\, pS_c - hH_c - cx_c \tag{3}$$

which has the same news vendor structure as the no-pooling case. The optimum stock level the supplier will carry is $F_c^{-1}(\frac{p-c}{p+h})$. Under this scenario, the supplier sets the optimum stock level disregarding any service level requirements the retailers may have.

3.3 Centralized Supply Chain Makes Pooling Decision

If both the retailers and the supplier were owned by the same company, the resulting centralized problem would be

$$\max\, (p_M + p)S_c - hH_c - cx_c \tag{4}$$

The centralized system revenue on each unit sold is $p + p_M$. Expression 4 has the form of a news vendor problem and so the optimal stock level is $F_c^{-1}(\frac{p+p_M-c}{p+p_M+h})$. The following observation relates the total stock in the centralized system to the total stock in the decentralized system where the supplier decides on the size of pooled inventory.

Observation 3.1 *In a decentralized system, the supplier always stocks less than the system-optimum stock level.*

Comparison of the critical ratio for the centralized system, $\frac{p+p_M-c}{p+p_M+h}$, with the critical ratio for the supplier, $\frac{p-c}{p+h}$, yields that $\frac{p+p_M-c}{p+p_M+h} > \frac{p-c}{p+h}$, which is equivalent to Observation 3.1. This is not surprising since it is the supplier who incurs the procurement and holding costs and thus has incentive to understock. This observation also indicates that the decentralized system will not reach its total sales capacity. On the other hand if the stock level is set to that of the centralized system under coordination, the supplier will profit less than she would in a decentralized system, where she can set inventory levels optimally.

Another important point is that $F_c^{-1}\left(\frac{p+p_M-c}{p+p_M+h}\right)$ maximizes total supply chain profit profit but may not satisfy the service level requirements for the retailers. This means that enforcement of service level requirements may decrease total system profit. We explore this observation in the next section.

3.4 Weak Supplier, Weak Retailers: Inventory Pooling Subject to Service Constraints

Consider a supply chain where the supplier is too weak to make the pooling decision by herself and the retailers are too weak to preclude pooling. Instead, the retailers allow the supplier to pool inventory subject to the service level constraints they set.

Because of competition, retailers may be willing to share some but not all inventory. Thus we may consider the total stock to be broken up into four partitions. The supplier holds two types of inventory for each retailer: shareable and reserved. Shareable inventory may be used to satisfy the other retailer's demand once the demand of the primary inventory owner is satisfied; whereas reserved inventory cannot. For example, if the stock kept for retailer 1 runs out and there is stock available only in the reserved section of the inventory for retailer 2 then this cannot be used to satisfy the unsatisfied demand of retailer 1. Let us define the notation:

$$\begin{aligned} x_i^d &= \text{amount of reserved stock for retailer } i \\ x_i^s &= \text{amount of shareable stock for retailer } i \end{aligned}$$

Total expected sales after pooling and total expected left-over inventory are simply the sum of the individual expected sales and expected left-over inventory figures. The problem of maximizing total profit may be formalized as

$$\begin{aligned} &\max\, pS_c - hH_c - c(x_1^d + x_2^d + x_1^s + x_2^s) \\ &\text{subject to} \\ &\rho_i \geq \underline{\rho}_i,\ \ i \in \{1, 2\} \end{aligned}$$

When cost structures are symmetric and there are no extra incentives/costs regarding inventory sharing, we make the following observation.

Observation 3.2 *To maximize total expected profit, one need never hold reserved inventory.*

This result is easy to see since the supplier's profit when $x_1^d = x_2^d = 0$ is at least as much as her profit when $x_1^d > 0$ and $x_2^d > 0$. A model similar to our 4-partition model allows only a fraction,

$f \leq 1$, of a retailer's demand to be met at another retailer (or in our case using his stocks). This restriction may be due to transshipment delays or a fraction of customers not willing to wait. This differs from our model in that if the extra demand at retailer i is large enough, regardless of how small f is, the spill-over demand can deplete all extra inventory at retailer j with positive probability. In our model, if $x_i^d > 0$ then whether it would be depleted or not depends only on the magnitude of demand at retailer i.

We drop the superscript notation differentiating between reserved and shareable inventory because by Observation 3.2 reserved inventory is zero in an optimal solution. Let D_i and D_j be the random variables representing the demand at retailers i and j respectively. Under this complete pooling scheme, the probability of no stock-out at retailer i is

$$\rho_i = P(D_i \leq x_i) + P(x_i \leq D_i \leq x_i + x_j - D_j) \tag{5}$$

In the remainder of this section, we concentrate on calculating stocking levels after pooling. We first analyze the supplier's problem and ignore the effects on the retailers. It is known that expected profit increases due to pooling. We would also expect total stock level to decrease. However, Gerchak and Mossman [16] give a simple example in which total inventory level after pooling is *higher* than the total inventory level before pooling.

When stocking levels increase, the expected service level provided to the retailers and their expected sales also increase. If the required service level exceeds the critical ratio, the supplier loses money by providing a higher service level. Therefore it is important to calculate the stock levels so that the service level constraints are binding whenever the service level requirements exceed the critical ratio. When we calculate stock levels in this way, we can show that stock levels after pooling do not exceed those before pooling as formalized in Lemma 3.1.

Lemma 3.1 *The after-pooling stock level does not exceed the total before-pooling stock level if the probability of no-stockout after pooling is equal to the probability of no-stockout before pooling for each retailer.*

Proof See Appendix B for all proofs.

3.4.1 Supplier-Optimal Pooled Stock Size

To avoid excessive inventory costs, the supplier should provide no more than the contracted service level when service level requirements are higher than the critical ratio. We make use of

	$\rho_2 \leq \rho_2^l$	$\rho_2^l < \rho_2 \leq \rho_2^u$	$\rho_1 > \rho_2^u$
$\rho_1 \leq \rho_1^l$	x_c^*	x_c^*	$F_2(x_2^*) = \rho_2, x_1^* = 0$
$\rho_1^l < \rho_1 \leq \rho_1^u$	x_c^*	Requires analysis (1)	Requires analysis (2)
$\rho_1 > \rho_1^u$	$F_1(x_1^*) = \rho_1, x_2^* = 0$	Requires analysis (2)	Solve service level equations

Table 1: Optimum pooled inventory level depending on service levels ρ_1 and ρ_2

this fact to characterize the optimal solution for the supplier in case of pooling subject to service constraints. The characterization also determines the sizes of x_1 and x_2, shareable stock over which retailers 1 and 2 have priority respectively after pooling.

Define $x_c^* = F_c^{-1}(\frac{p-c}{p+h})$, the optimum pooled inventory in the absence of service level constraints. Even though we assume complete sharing of available stock by the two retailers, we still distinguish the levels, x_1 and x_2, over which retailers 1 and 2 have priority in case of a stockout, because these levels determine the respective service levels observed at the retailers. By letting $x_1 = x_c^*$ or $x_2 = x_c^*$, we can obtain the boundary values on service level at the two retailers. Further define for $i, j \in \{1, 2\}$

$$\begin{aligned} \rho_i^l &= \text{service level at retailer } i \text{ when } x_i = 0 \text{ and } x_j = x_c^* \\ \rho_i^u &= \text{service level at retailer } i \text{ when } x_i = x_c^* \text{ and } x_j = 0 \end{aligned}$$

With respect to these boundary values, the required service level pair (ρ_1, ρ_2) will fall in one of the nine regions depicted in Table 1. For three of the nine combinations, x_c^* is also a feasible total stocking level given the service level requirements. For the two cases, in which one requirement is below its corresponding lower bound and the other is above its corresponding upper bound, the optimal stocking level is found by solving the service level constraint for the higher service level and setting the other stocking level to zero. In this case, the retailer with the lower service level has no stock over which he has priority. The stock kept for the retailer with the higher service level is used to cover the other retailer. This situation, although optimal for the supplier, may create a conflict of interest between the retailers and therefore may be unacceptable because the retailer with the higher service level requirement is underwriting the service level of the other retailer. For the case where both service level requirements exceed their corresponding upper bounds, the stocking level is found by solving both of the service level constraints as equalities. The solution is optimal because it provides the least stock to satisfy both of the equations.

If the service level pair falls in the region marked by (1), the situation is more complicated: If $F_c^{-1}(\frac{p-c}{p+h})$ can be partitioned such that both of the constraints are satisfied then it is obviously the optimal stock level. This can be checked simply by finding the partition that would still satisfy the service level constraint at the retailer with the higher requirement and then verifying whether the same partition satisfies the service level constraint of the other retailer. If so, $F_c^{-1}(\frac{p-c}{p+h})$ is the optimal stocking level. If not, the second step is to set $x_i^* = 0$ where i is the retailer with the lower service level requirement and find x_j^* that satisfies retailer j's service level constraint. If x_j^* also satisfies retailer i's constraint then it is optimal, as established in Lemma 3.2. Otherwise, one needs to solve for x_i^* and x_j^* by setting the two service level constraints as equalities. Clearly, providing more service (higher stock levels) is suboptimal.

Lemma 3.2 *When $F_c^{-1}(\frac{p-c}{p+h})$ is not feasible, $x_c^* = x_j^* = F_j^{-1}(\rho_j)$, where j is the retailer with the higher service level, is optimal when it is feasible.*

If the the service level pair falls in the region marked by (2) then first find $x_j^* = F_j^{-1}(\rho_j)$ where j again is the retailer requiring the higher service level. If x_j^* is also feasible for retailer i then $x_j^* = x_c^*$ is the optimal stock level. If not, one needs to solve for x_i^* and x_j^* by setting the two service level constraints as equalities as in the case of (1).

3.4.2 Retailer Profits under Pooling

Retailer profits may decrease due to pooling because the total inventory in the supply chain decreases. This phenomenon was first observed by Anupindi and Bassok [1] in a different setting, where the retailers pay the holding cost and it is their decision whether to pool inventory or not. We show by example that this loss cannot be prevented even with the introduction of Type-1 service measure constraints.

Example 3.1 *Consider a system with two retailers. Let both demand distributions be $U(0,1)$ and the critical ratio be $\frac{p-c}{p+h} = 0.9$. Then before pooling, the optimal stocking levels are $x_1 = x_2 = 0.9$ with total expected sales at 0.99. The before-pooling service levels at the retailers are each 0.9. The stock level corresponding to $F_c^{-1}(\frac{p-c}{p+h})$ is 1.55279. Using equation 5 and letting $x_1 = x_2 = 1.55279/2$, this stock level corresponds to a service level of approximately 0.92 at each of the retailers, which means service level constraints are more than satisfied. However, the total expected sales is 0.985. Therefore, the expected profits of the retailers drops even though the service level constraints are satisfied.*

Thus a simple contract between the retailers and the supplier, where the retailers only enforce their expected service levels, is not adequate to protect the retailers from losing sales when the supplier has the power to pool inventory.

In Appendix A we briefly discuss another service measure that can guarantee profits for the retailers; but one not frequently used because it is hard to measure. Like most other researchers we use the easier measure of service, probability of no-stockout; but we compensate to some extent for its deficiencies by proposing a profit allocation mechanism that ensures expected profits of all parties involved in the contract remain at before-pooling levels.

4 Coalitions in Cooperative Games and Shapley Value

We analyze the inventory pooling problem among the retailers and the supplier as a cooperative game, which allows for the possibility of coalitions among players. Coalitions are possible because players are assumed to negotiate effectively with each other [20]. Let $N = 1, 2, \ldots, n$ be the set of players. For each coalition, $J \subseteq N$, of supply chain partners let the value of the coalition $v(J)$ be the total expected profit of coalition J. For each coalition J, $v(J)$ consists of two parts: the total expected profit of the retailers and the supplier in the coalition and the total profit the supplier earns due to the retailers who are not in the coalition. By definition, $v(\emptyset) = 0$. We use the subscript notation to represent the elements of set J; that is if $J = \{1, 2, S\}$, $v(J) = v_{12S}$ denotes the expected profit of a coalition consisting of retailers 1 and 2 and the supplier, denoted by S.

An allocation ϕ is a vector, where each ϕ_i is the payoff to player i. Given that N represents the grand coalition, an allocation ϕ is said to be in the core of v if and only if

$$\begin{aligned} \textstyle\sum_{i\epsilon N} \phi_i &= v(N) \\ \textstyle\sum_{i\epsilon J} \phi_i &\geq v(J), \forall J \subseteq N \end{aligned}$$

If an allocation is not in the core there is incentive for some players to leave the coalition. A core solution is desirable because it is stable; but the core of a cooperative game may be empty. In addition, even when the core exists, an allocation in the core may have other undesirable characteristics. For example, it may be extreme and/or sensitive to system parameters (Myerson [20, page 429]) or may fail to satisfy coalitional monotonicity (Granot and Sosic [17]). In general, it is hard to determine whether the core of a coalitional game exists or not. Even when it does, the more important question is whether the suggested value allocation scheme is actually in the core. While such issues can be important, we avoid them as unpromising in this context. Instead, we follow Shapley [29] in representing the expected payoff to player i, $\phi_i(v)$, as the unique solution to the following axioms. For the second axiom, a *carrier* of v is any set $U \subseteq N$ with $v(S) = V(U \cap S)$, $\forall S \subseteq N$.

- **Symmetry** For all permutations $\Pi(N)$ of N, $\phi_{\pi i}(\pi v) = \phi_i(v)$ for each permutation π in $\Pi(N)$.
- **Efficiency** For each carrier U of v $\sum_U \phi_i(v) = v(U)$.
- **Law of aggregation** $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$.

These axioms are meaningful and practical in terms of our problem. We would expect players of equal power to receive the same allocation and the first axiom ensures that the Shapley value allocation only depends on the contribution of the player to the coalitions. The second axiom makes sure that the Shapley value allocation mechanism allots the total worth of the coalition to the players and a player who is not in the carrier receives zero allocation. Again, in our context we would expect any reasonable allocation mechanism to exhaustively distribute the total profit of the system to the players and to assign zero value to a player who does not increase the value of a coalition. Finally, if the players play two different games with value functions v and w, then the total Shapley value allocation to player i is the same as if the players were to play a game with value function $v + w$. This axiom shows that Shapley value allocations are not dependent on the time of bargaining between the players.

The Shapley value as stated in Expression 6 may be interpreted as the expected marginal contribution of player i to a coalition. In Expression 6, the term $(v(J \cup \{i\}) - v(J))$ is the marginal contribution of player i to coalition J. We can interpret the fractional term as follows. There are $|N|!$ different ways all the players are ordered to enter the grand coalition and $\mid J \mid!(\mid N \mid - \mid J \mid -1)!$ different ways all the players in J enter the grand coalition before player i does. Assuming all orderings are equally likely, $\frac{|J|!(|N|-|J|-1)!}{|N|!}$ is the probability a coalition J is already formed before i enters the coalition (for a more detailed interpretation see Myerson [20]).

$$\phi_i(v) = \sum_{J \subseteq N_1} \frac{\mid J \mid!(\mid N \mid - \mid J \mid -1)!}{\mid N \mid!} (v(J \cup \{i\}) - v(J)) \tag{6}$$

In the inventory centralization context, coalitions are formed when a subset of players agree to pool inventory. We propose a value-sharing mechanism where each player's after-pooling profit allocation is equal to his Shapley value.

5 Shapley Value Allocations for Two-Retailer Games

For two retailers and one supplier, the value of the coalition increases only when all three players agree to inventory pooling. Therefore, the value of a 2-player coalition is the sum of the individual expected profits of the players before pooling. This simplifies the calculation of the Shapley value for player i ($i \in \{1, 2, S\}$) to

$$\phi_i(v) = \frac{2}{3}\, v_i + \frac{1}{3}\left(v_{12S} - \sum_{j \in \{1,2,S\}, j \neq i} v_j\right) \tag{7}$$

where v_{12S} is the value of the coalition when all three players agree to pooling and v_1,v_2, and v_S are the individual expected profits of the players before pooling. Equation 7 tells us that in the Shapley value allocation, for each player i, the weight of his contribution to the coalition is half the weight of his before-coalition payoff. The Shapley value formalizes the rule for the allocation of total profit to the three players. However to fully characterize the value-sharing mechanism we also need to define a rule for calculating the individual expected profits of the players without pooling. Without pooling, the supply chain has the structure described in Section 3.1. If the retailers do not agree to pooling under the Shapley value allocation rule, they will be reserved a stock level of $F_i^{-1}\left(\max\left(\underline{\rho}_i, \frac{p-c}{p+h}\right)\right)$. Therefore v_1, v_2, v_S are calculated with respect to the stock levels set at $F_i^{-1}\left(\max\left(\underline{\rho}_i, \frac{p-c}{p+h}\right)\right)$ for each retailer.

Writing Expression 7 in a different way, we obtain the equivalent expression

$$\phi_i(v) = v_i + \frac{1}{3}\left(v_{12S} - v_1 - v_2 - v_S\right) \tag{8}$$

which shows that for two retailers, the three players share the extra revenue due to pooling equally. Each player's expected payoff is his expected payoff before pooling plus one third of the increase in total expected system profit due to pooling.

We next establish some stability properties of the Shapley value allocations.

Theorem 5.1 *The Shapley value allocation scheme induces coordination of the supply chain.*

An allocation for player i is *individually rational* if it is at least as much as what the player would get if he had not participated in the coalition, that is $\phi_i(v) \geq v(\{i\})$.

Proposition 5.1 *The Shapley value allocations for the inventory holding game are individually rational for all of the players.*

The next proposition shows that the Shapley value allocations are in the core of the game and thus establishes that the core of the game is non-empty.

Proposition 5.2 *The Shapley value allocations are in the core of the inventory holding game.*

Thus when the Shapley value is used as the profit allocation scheme in a 2-retailer supply chain, the retailers and the supplier have incentive to form pooling coalitions. In addition, the resulting coalition is stable (in the core) and the total joint profit is the maximum the supply chain can attain.

6 Second-Order Instabilities

That the profit allocations under Shapley value allocation scheme are individually rational and in the core may not be adequate to prevent what we call *second-order instabilities.* These kinds of instabilities may arise if one or more of the players believe there is asymmetric, unfair profit allocation to some other player(s). In cooperative game theory, it is assumed that players would not be willing to deviate from coalitions if individual rationality constraints are satisfied and the allocations are in the core. However, players may hesitate to form coalitions if they believe their competitor benefits more than he should from the coalition. They may require further adjustments to the coalition contract, for example in the form of side payments.

In the remainder of this paper we use the BP and AP notation in the superscript to differentiate the values each variable (such as inventory level, expected sales) takes before pooling and after pooling respectively.

6.1 Shapley Value Allocations Favor Retailers

Retailer profit is the product of sales by the mark-up per item and so we define *effective sales* at retailer i as the Shapley value allocation to retailer i divided by the unit mark-up, and

$$\text{E[effective sales at retailer } i] = \frac{\phi_i}{p_M}$$

Comparing total expected effective sales by total expected actual sales after pooling, we can determine whether the retailers get more than their contribution to total after-pooling profit, in which case the supplier gets less than her contribution. More specifically, we are interested in knowing when the following inequality occurs:

$$\text{E[total effective sales]} = \frac{\phi_1 + \phi_2}{p_M} > \text{E[total sales after pooling]} \tag{9}$$

Theorem 6.1 *Total retailer allocations are greater than actual retailer contribution to after-pooling profit if and only if the expected change in supplier profit exceeds the expected change in average retailer profit.*

In other words, when the change in expected profit for the supplier after pooling is greater than the average change for the retailers, the supplier is forced to give up a portion of her extra profits to the retailers, the size of which is determined by the Shapley value calculations. **Proof**

Even when Expression 9 holds, it is possible that only one of the retailers benefits from the extra allocation:

Example 6.1 *Consider two retailers with iid* $U(0,1)$ *demand. Service level is set at* 0.9 *by retailer 1 and at 0.65 by retailer 2. Let* $p = 4$, $p_M = 4$, $c = 2$, *and* $h = 0.1$. *The ex-post profit allocations are:* $\phi_1 = 2.367776$ *and* $\phi_2 = 1.911776$. *E[total sales after pooling]* $= 0.980813$ *and E[total effective sales] is* $(2.367776 + 1.911776)/4 = 1.069888$. *Comparing the two,* $1.069888 > 0.980813$ *implies that the retailers' total allocation is greater than their total expected profit. In addition, the effective sales for retailer 2 is* $1.911776/4 = 0.477944$. *However,* $0.980813 - 0.477944 > 0.5$, *which implies his effective sales is less than his expected sales (because expected sales at retailer 1 cannot exceed 0.5). Therefore retailer 2's allocation under Shapley value scheme is less than his expected sales revenue after pooling.*

In this example both retailer 2 and the supplier get allocations less than their individual contributions to total after pooling profit, while retailer 1 gets a higher allocation. In this example, this is a fair allocation because retailer 1 requests a higher service level before pooling. Retailer 2, by forming a pooling coalition with retailer 1, gains access to a larger stock but has to to give up some of his profits to retailer 1.

Define the following notation for ease of presentation. Let Λ be the change in the supplier's expected cost and Δ_i be the change in expected sales at retailer i due to pooling.

$$\begin{aligned} \Lambda &= c\left(x_1^{BP} + x_2^{BP} - x_1^{AP} - x_2^{AP}\right) + h\left(H_1^{BP} + H_2^{BP} - H_1^{AP} - H_2^{AP}\right) \\ \Delta_i &= S_i^{AP} - S_i^{BP}, \quad i \in \{1, 2\} \end{aligned}$$

Proposition 6.1 *Given E[total effective sales] $\geq$ E[total sales after pooling], if the change in expected sales at retailer i is greater than or equal to the change in expected sales at retailer j then E[effective sales at retailer j] $\geq$ E[sales at retailer j after pooling].*

Proposition 6.1 says that the expected change in retailer i's sales after pooling is greater than the change in retailer j's sales ensures that retailer j's final profit allocation will correspond to an effective sales level higher than his expected sales. However the same condition is not adequate to ensure the same for retailer i. This result is counterintuitive because we would normally expect retailer i would be ensured a greater portion of the extra profit due to pooling since he is making the more positive impact on expected sales.

The Shapley value allocation rule, since it is in the core, guarantees that none of the supply chain players can be better off by breaking away from the coalition. However, while one player may be only infinitesimally better off when compared to the no-pooling scenario, another player may receive a significantly high allocation, an allocation that is more than that player's contribution to total supply chain profit. This inequitable distribution of savings is in the core and so is stable in a technical sense. But many people would find it well within the range of human behavior for the player receiving the lower allocation to refrain from pooling and forgo his minuscule extra profits. This illustrates a weakness of the concept of "core".

7 With Whom to Form a Coalition?

In the previous section, we have shown that even though the Shapley value allocation scheme ensures profit allocations higher than before-pooling profit levels for all players, some players may get more favorable allocations. Therefore it is important for all players to know with whom it is most advantageous to form pooling coalitions. In this section, we analyze this question from the points of view of the retailers and the supplier separately. We take required service level and the demand distribution as the defining characteristics of the retailers. Cost and revenue parameters are still assumed to be identical for both of the retailers.

7.1 The Retailer's Perspective

The question we seek to answer is: "Given a fixed service level for retailer i, at what service level for retailer j would retailer i form a coalition with retailer j?" Throughout this section we make use of the following rule in the contract: before-pooling profit levels, v_i, v_j, and v_S are calculated with respect to the stock levels set at $F_i^{-1}\left(\max\left(\underline{\rho}_i, \frac{p-c}{p+h}\right)\right)$ for each retailer. Therefore, our region of interest is $\rho_j \in \left(\frac{p-c}{p+h}, 1\right)$ because in the region $\left(0, \frac{p-c}{p+h}\right]$ the stock level is set at $F_j^{-1}(\frac{p-c}{p+h})$ regardless of the service level requirement. When the stock level for retailer j is fixed at $F_j^{-1}(\frac{p-c}{p+h})$, the service level requirement of retailer j does not have an impact on the ex-post profit allocation to retailer i. The following theorem establishes that the profit allocation to one retailer is unimodal in the service level requirement of the other retailer.

Theorem 7.1 *The Shapley value profit allocation to retailer i is a unimodal function of service level ρ_j of retailer j. In addition, $\rho_j^* = \frac{p+p_M-c}{p+p_M+h}$ is the global minimizer of the payoff to retailer i.*

In all examples we studied, $\phi_i(\rho_j)$ has always been a convex function. However, we could not prove this in general because $\frac{\partial^2 \phi_i(\rho_j)}{\partial \rho_j^2}$ is a function of $\frac{\partial^2 F_j^{-1}(\rho_j)}{\partial \rho_j^2}$, which is difficult to sign. However, proving unimodality is sufficient for our purposes because the interesting point in this theorem is that the ex-post profit allocation to a retailer decreases if he forms a coalition with a retailer with service level in the range $(\frac{p-c}{p+h}, \frac{p+p_M-c}{p+p_M+h})$.

The next natural question is whether there is a threshold service level ρ_j in the region $(\frac{p+p_M-c}{p+p_M+h}, 1)$ beyond which $\phi_i(\rho_j)$ is greater than $\phi_i(\frac{p-c}{p+h})$. The answer is "not necessarily".

Proposition 7.1 *When the demand distribution for retailer j has infinite support, then the ex-post profit allocation for retailer i goes to infinity as ρ_j goes to 1.*

Thus when $F_j(\cdot)$ has infinite support there is a range of ρ_j beyond $\frac{p+p_M-c}{p+p_M+h}$, where $\phi_i(\rho_j)$ is greater than $\phi_i(\frac{p-c}{p+h})$, and retailer i always prefers to form a pooling coalition with a retailer requiring a high service level. However, when $F_j(\cdot)$ has finite support, whether such a region exits or not depends on the system parameters as we demonstrate with the following example.

Example 7.1 *Let the demand function for retailer 2 be U(0,1). The demand function for retailer 1 is arbitrary but independent from that of retailer 2. Let $\rho_1 = 0.96, p = 5, c = 2, h = 0.1, p_M = 5.5$.*

In Figure 2: Case 1, the highest value $\phi_1(\rho_2)$ attains beyond $\frac{p+p_M-c}{p+p_M+h}$ is still lower than $\phi_1(\frac{p-c}{p+h})$. However, if we change p_M to 2, Figure 2: Case 2 shows that higher profit allocations are possible for retailer 1 beyond $\frac{p+p_M-c}{p+p_M+h}$.

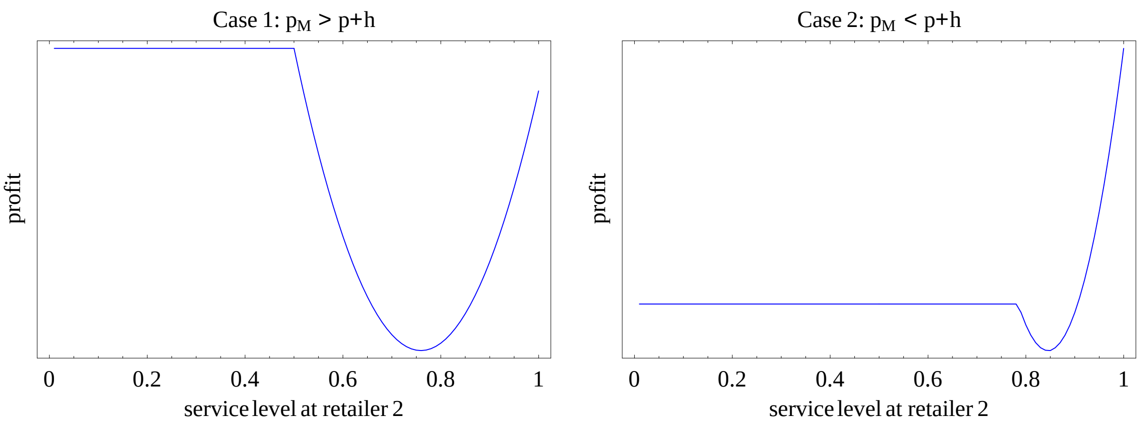

Figure 2: Profit allocations to retailer 1 as retailer 2's service changes (graphs not to scale)

If the demand distribution of retailer 2 is U(0,1), $\lim_{\rho_2 \to 1} \phi_1(\rho_2) > \phi_1(\frac{p-c}{p+h})$ when $p_M < p+h$. This condition does not depend on the value of c. We can interpret this result if we consider p_M to be the potential profit to the whole supply chain from the sale of a single item and $p+h$ to be the potential loss to the supplier when an item does not sell. When the potential loss to the supplier is large, she will tend to under-stock and this hurts the retailers. However, when the service level requirement of one or both of the retailers is very high, the supplier will have to stock enough to cover the requirement even if it is suboptimal for herself. Therefore, when the overage cost is very high, it is better for a retailer to form a coalition with a retailer with a high service level requirement since this would force the supplier to stock more.

The retailers share their minimum service level requirements with the supplier and not necessarily with each other. However a retailer may still infer information regarding the service levels at other retailers by observing other properties such as small versus large retailer, small versus large market share. The retailers can differentiate more favorable pooling partners based on this type of prediction of service level requirements.

7.2 The Supplier's Perspective

In section 3.1 we set the contract such that the before-pooling profits are calculated to maximize the before-pooling supplier profit as long as the service level constraints set by the retailers are satisfied. This means that the before-pooling inventory levels are calculated as $F_i^{-1}(\max(\underline{\rho_i}, \frac{p-c}{p+h}))$ for each retailer i. Although this maximizes the supplier profit before pooling and guarantees at least ρ_i level of service for each retailer, this calculation may not maximize the supplier's after-pooling profit according to the Shapley value allocation scheme. The next theorem shows that the Shapley value allocation to the supplier is a unimodal function of the service level requirements of the retailers. Figure 3 is an example of how supplier profit changes as the service level requirement of one of the retailers changes.

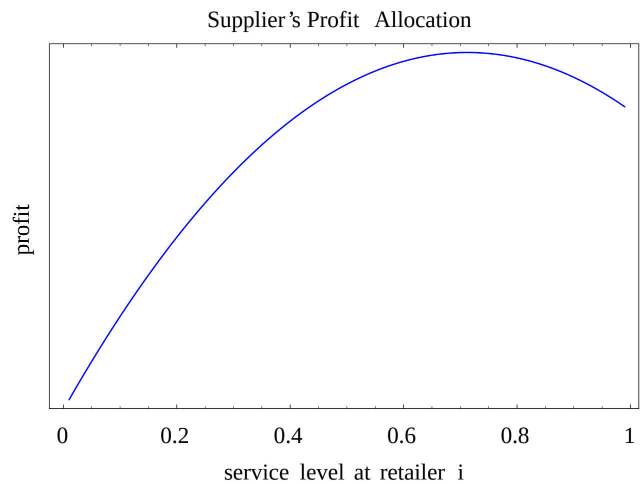

Figure 3: Supplier profit allocation as a function of service level

Theorem 7.2 *The Shapley value allocation to the supplier is unimodal in the service level requirements of the retailers and the global maximum occurs at*

i. $(\rho_1, \rho_2) = \left(\max\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right), \max\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)\right)$ *if* $\frac{2(p-c)-p_M}{2(p+h)-p_M} < 1$,

ii. $(\rho_1, \rho_2) = (0, 0)$ *otherwise.*

Theorem 7.2 states that the supplier has incentive to relax the terms of the contract. The current contract calculates before-pooling profits using $x_i = F_i^{-1}(\max(\underline{\rho_i}, \frac{p-c}{p+h}))$ for each retailer. Hence the supplier guarantees each retailer a service level of at least $\frac{p-c}{p+h}$, which is higher than the service level that maximizes her Shapley value allocation. Therefore the supplier prefers a contract that calculates before-pooling profits based on $x_i = F_i^{-1}(\underline{\rho_i})$ — a contract that does not place a lower bound on the service she provides. Then she is allowed to maximize her after-pooling profits by setting the service level at $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$ for the retailer(s) requiring a service level that is less than or equal to $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$. Unless at least one of the retailers requires a service level smaller than $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$, the supplier does not have room for manipulation since the contract still guarantees that the after-pooling profit allocations are at least as much as the before-pooling profits (as set through the service level constraint $\underline{\rho_i}$).

7.3 Conflict Between Retailers and Supplier

In the previous two sections, we looked at how service level requirements can be used to optimize profits by both the retailers and the supplier. However, we did not analyze the effects of these decisions on the other parties in the coalition. The total supply chain profit does not increase when the supplier maximizes her profits by varying the terms of the contract and relaxing the lower bound on service. Therefore the Shapley value allocation to one or both of the retailers must be reduced. We would like to know "what happens to the profits of the retailers when the supplier maximizes her profit?".

We define the *base case* as the case where the stocking levels are determined by $F^{-1}(\frac{p-c}{p+h})$. From Theorem 7.2 we know that supplier profit is maximized at either $(\rho_1, \rho_2) = (0, 0)$ or $(\rho_1, \rho_2) = \left(\max\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right), \max\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)\right)$. Clearly $(\rho_1, \rho_2) = (0, 0)$ is not implementable. Therefore the supplier wants to set $(\rho_1, \rho_2) = \left(\frac{2(p-c)-p_M}{2(p+h)-p_M}, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$ and this requires $p - c \geq p_M/2$. The supplier's per-unit profit $(p - c)$ needs to be at least as much as half the retailers' total per-unit profit (p_M) for the supplier to be able to maximize her after-pooling profits. We can interpret this condition as a measure of the relative power of the supplier. If the supplier is making a high per-unit margin on each item she sells, she has the ability to manipulate the contracted service levels whenever the retailer requirements allow it.

For two random variables X and Y with distribution functions $F(\cdot)$ and $G(\cdot)$, X is said to be larger than Y in *dispersive order* if $F^{-1}(\beta) - F^{-1}(\alpha) \geq G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 < \alpha \leq \beta < 1$ (denoted as $X \geq_{disp} Y$) (Shaked and Shanthikumar [28]). Dispersive order requires the difference between two quantiles of X_i to be smaller than the difference between the corresponding quantiles of X_j; therefore dispersive order compares the variability of the two distributions. Assuming there is dispersive order between the demand distributions, the following theorem identifies which one of the retailers (if either) will be better off when compared to the base case.

Theorem 7.3 *Assume $D_i \geq_{disp} D_j$. When the supplier maximizes her own after-pooling profit allocation by changing (ρ_i, ρ_j), either the after-pooling profit allocations to both of the retailers are reduced or the profit allocation to the one with smaller demand in dispersive order is increased while the profit allocation to the other is reduced when compared to the allocations under the base* *case.*

The next result directly follows from Theorem 7.3 since for two random variables Y and Z, $Y \leq_{disp} Z$ implies $\mathrm{Var}(Y) \leq \mathrm{Var}(Z)$.

Corollary 7.1 *If the demand of one retailer is greater than the demand of the other retailer in dispersive order and the supplier maximizes her own after-pooling profit allocation, either the profit allocation to the retailer with the smaller demand variance will increase or the profit allocations to both of the retailers will decrease when compared to the allocations under the base case.*

This result is intuitive in terms of the supply chain because when the supplier maximizes her profits, if the Shapley value allocation to one of the retailers will increase then it will be the one with smaller demand variance. This result is not surprising because the retailer with the smaller demand variance brings less risk into the pooling coalition and we would expect that retailer to receive a higher allocation.

The next theorem states that convolutions of random variables with logconcave densities can be ordered in the dispersive sense. This result implies that assuming dispersive order between the demand variables is not very restrictive.

Theorem 7.4 (Shaked and Shanthikumar [28], Theorem 2.B.3, p 71) *The random variable X satisfies $X \leq_{disp} X + Y$ for any random variable Y independent of X if and only if X has a logconcave density.*

Normal and gamma (with $p \geq 1$) distributions are frequently invoked models of demand distributions and they have logconcave densities [4]. Therefore, by Theorem 7.4 normal and gamma demands with different shape parameters can be ordered in the dispersive sense and thus satisfy the condition on Theorem 7.3.

Another interesting property of the dispersive order is $X \leq_{disp} Y$ if and only if $X + c \leq_{disp} Y$ for any real number c. This means that the dispersive order between two random variables is preserved even if there is a shift in the mean(s). This property has an interesting implication on our results: the retailer whose profits decrease due to the supplier maximizing her profits cannot reverse the situation (become the retailer whose profits increase) even if his mean demand increases and thus creates more sales. However he can reverse the situation by changing the shape of his demand distribution by reducing the demand variance, because the allocation mechanism favors the retailer with lower risk. Note that the coefficient of variation of $X + c$ is less than that of X when c is positive (that is when mean demand increases). The surprising result is that the ordering of the variances rather than that of the coefficients of variation determines which retailer is more likely to lose profits.

8 Conclusions

In an interesting recent survey on game theory as a tool in supply chain analysis, Cachon and Netessine emphasize that cooperative game theory has not received much attention in the supply chain literature in spite of its potential usefulness [10]. In the same chapter, Cachon and Netessine also indicate that the Shapley value has not yet been employed in supply chain research in spite of its desirable characteristics such as uniqueness. Robinson [26] and Hartman and Dror [18] consider Shapley value as a cost-allocation scheme but do not analyze the operational implications of using it. Granot and Sosic [17] appear to have been the first to mention Shapley value as a profit-allocation mechanism that may induce supply-chain-optimal inventory decisions but, as far as we know, this idea has not been followed up. We offer the present paper as an initial step in understanding the uses of Shapley value as a value-sharing mechanism to affect the operational decisions of supply chain partners.

Our model shares some limitations with most work in this area. For example, like others [1, 27, 31], we are limited by analytic tractability mostly to 2-retailers. In other work we have been able to extend some analysis to arbitrary numbers of retailers [7]. Similarly, to derive more particular results we have to make some simplifying assumptions about the demand distributions experienced by the retailers. We also assume that the service levels are determined exogenously to the cooperative game. They are either industry-driven or set through negotiations that are beyond the scope of our model. This assumption allows us to ignore incentives to set service levels strategically.

We have analyzed the behaviors of the supply chain members under the proposed value-sharing mechanism. It is important to compare various mechanisms for coordinating the supply chain by studying the strategic behavior that they might induce. For example, how will supply chain players answer such questions as with whom to form a coalition or whether one can game the system?

We are assuming a long-term relationship among the supply chain partners because we model the pooling problem as an *allocation game in expectation* (AGE) [3]. Another approach is a *snapshot allocation game* (SAG), which is used by Anupindi et al. [2]. In SAG, the value of the game is calculated based on each realization of random demand. While allocations in the core of SAG are renegotiation proof, allocations for AGE implicitly assume the players will not break from the contract based on individual realizations of demand [3].

The Shapley value allocations for the 2-retailer supply chain correspond to equal sharing of extra revenue due to pooling. Cachon and Lariviere [9] analyze revenue-sharing contracts and identify their limitations. They conclude that revenue sharing is not prevalent in practice partly because of high administrative costs and difficulties in monitoring revenues of retailers. Similar shortcomings apply to our value-sharing mechanism as well. We are proposing a contract where the three players first pool their profits and then the total is redistributed to them according to the Shapley values. We can think of this as a taxing mechanism where some players pay their taxes (return some of their profit) and some players get refunds (receive payments). This framework would work best if the supply chain members are in a long-term relationship, which is also the implicit assumption underlying AGE. All members are better off pooling inventory and sharing it based on Shapley value; however the mechanism will not work if there is doubt some player will break away from the coalition after getting a refund and will not be there to pay his tax when it is his turn.

As Cachon and Lariviere [9] emphasize, to share value, it must be possible to monitor revenues of the retailers. The Shapley-value mechanism, in addition, requires visibility of both the stocking level of the supplier and her costs. Our proposed value-sharing mechanism also raises the issue of information guessing at the retailers: Can players infer information about their coalition partners that might allow them to gain advantages? To answer this and similar questions we plan further research on the truth-inducing properties of our model.

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Appendix A: Service Contracts and Fill Rate as an Alternative Service Measure

A service contract based on probability of no stock-out does not always guarantee profits for the retailers. Although a more sophisticated service contract based on *fill rate* can achieve this, fill rate has weaknesses that render it less attractive as a basis for sharing than Shapley value. Fill rate β is defined as the fraction of demand routinely satisfied from shelf:

$$\beta = 1 - \frac{\text{E[shortage]}}{\text{E[demand]}}$$

We also differentiate between fill rate observed at retailer i before and after pooling, β_i^b and β_i^a respectively. Define $E[x_{ji}]$, the expected size of retailer j's shareable stock used by retailer i. Expected before and after-pooling fill rates are

$$\begin{aligned}
\beta_i^b &= 1 - \frac{\int_{x_i}^{\infty}(y_i-x_i)f_i(y_i)dy_i}{\mu_i} \\
&= \frac{S_i^b}{\mu_i} \\
\beta_i^a &= 1 - \frac{\int_{x_i}^{\infty}\int_{x_j}^{\infty}(y_i-x_i)f_i(y_i)dy_i f_j(y_j)dy_j + \int_0^{x_j}\int_{x_i+x_j-y_i}^{\infty}(y_j-(x_i+x_j-y_i))f_i(y_i)dy_i f_j(y_j)dy_j}{\mu_i} \\
&= \frac{S_i^b + \int_0^{x_j}(1-F_i(x_i+x_j-y_i))F_j(y_i)dy_i}{\mu_i} \\
&= \frac{S_i^b + E[x_{ji}]}{\mu_i} \\
&= \frac{S_i^a}{\mu_i}
\end{aligned}$$

Expected fill rate is a function of expected sales when unsatisfied demand is lost. Therefore contracting to assure a minimum expected fill rate guarantees a minimum expected sales level, and thus a minimum profit level, for the retailers. In addition, after pooling, fill rate at retailer i increases by the expected size of retailer j's shareable stock used by retailer i scaled by expected demand.

Fill rate, but not probability of no stock-out, ensures minimum expected sales because fill rate takes into account the size of a shortage when it happens, whereas probability of no stock-out does not. The size of shortages becomes important in evaluating expected sales when sales are lost in case of a stock-out. In addition, the magnitude of probability of stock-out is not a good estimate of the ratio of unsatisfied demand to expected demand [24].

Even though a service contract based on fill rate guarantees a minimum profit level for the retailers in case the supplier pools inventory, it does not necessarily induce the supplier to hold the supply-chain-optimal level of inventory. In addition, due to the dependencies in service levels after pooling, calculations become complicated especially as the number of retailers in the supply chain increases. Therefore we find the Shapley value allocation mechanism to be more useful than a service contract based on fill rate.

Appendix B: Proofs

Proof of Lemma 3.1 We find the before-pooling inventory levels x_1 and x_2 as solutions to

$$\rho_i = F_i(x_i) \qquad i = 1, 2 \tag{10}$$

Then defining x_1' and x_2' as the after pooling inventory levels and using Equation 5, we obtain the following two equations.

$$\begin{aligned}
\rho_1 &= F_1(x_1') + P(x_1' \leq D_1 \leq x_1' + x_2' - D_2 \text{ and } D_2 \leq x_2') \\
&= F_1(x_1') + \int_0^{x_2'} \int_{x_1'}^{x_1'+x_2'-y_2} f_1(y_1)\, f_2(y_2)\, dy_2\, dy_1
\end{aligned}$$

$$\begin{aligned}
\rho_2 &= F_2(x_2') + P(x_1' \leq D_2 \leq x_1' + x_2' - D_1 \text{ and } D_1 \leq x_1') \\
&= F_2(x_2') + \int_0^{x_1'} \int_{x_2'}^{x_2'+x_1'-y_1} f_2(y_2)\, f_1(y_1)\, dy_1\, dy_2
\end{aligned}$$

For each of these equations, the second term is clearly greater than or equal to zero. By Expression 10 and the fact that $F_1(\cdot)$ and $F_2(\cdot)$ are non-decreasing functions of inventory level $x_1' \leq x_1$ and $x_2' \leq x_2$, which proves the claim. □

Proof of Lemma 3.2 Since the supplier profit is maximized beyond $\frac{p-c}{p+h}$ for the smallest stock level that satisfies the service level constraints, all we need to show is that $x_j^* = F_j^{-1}(\rho_j), x_i^* = 0$ gives a smaller inventory level than having both $x_j^* > 0, x_i^* > 0$. Consider two cases. In the following proof, we use the additional 1 or 2 in the subscript to denote the inventory levels under cases 1 and 2 respectively.

Case 1: Let $x_{i1} = 0$. The inventory level pair (x_{i1}, x_{j1}) are set so as to satisfy the service level constraints. The service level expressions are:

$$\begin{aligned}
F_j(x_{j1}) &= \rho_j \\
\int_0^{x_{j1}} F_i(x_{j1} - y_j)\, f_j(y_j)\, dy_j &\geq \rho_i
\end{aligned} \tag{11}$$

Case 2: Let $x_{i2} > 0$. The corresponding service level expressions are:

$$F_j(x_{j2}) + \int_0^{x_{i2}} F_j(x_{i2} + x_{j2} - y_i)\, f_i(y_i)\, dy_i - F_i(x_{i2})F_j(x_{j2}) = \rho_j \tag{12}$$

$$F_i(x_{i2}) + \int_0^{x_{j2}} F_i(x_{i2} + x_{j2} - y_j)\, f_j(y_j)\, dy_j - F_i(x_{i2})F_j(x_{j2}) \quad \geq \quad \rho_i$$

The assumption $x_{i2} \geq 0$ implies $\int_0^{x_{i2}} F_j(x_{i2}+x_{j2}-y_i)\, f_i(y_i)\, dy_i - F_i(x_{i2})F_j(x_{j2}) \geq 0$. Therefore $x_{j1} \geq x_{j2}$. Now let $x_{i2} = x_{j1} - x_{j2}$ and compare the left hand sides of Equations 11 and 12.

$$\begin{aligned} & F_j(x_{j2}) + \int_0^{x_{j1}-x_{j2}} F_j(x_{j1} - y_i)\, f_i(y_i)\, dy_i - F_i(x_{j1} - x_{j2})F_j(x_{j2}) \\ \leq \quad & F_j(x_{j2}) + (F_j(x_{j1}) - F_j(x_{j2}))\, F_i(x_{j1} - x_{j2}) \\ = \quad & F_j(x_{j1})\ F_i(x_{j1} - x_{j2}) + F_j(x_{j2})\ (1 - F_i(x_{j1} - x_{j2})) \\ \leq \quad & F_j(x_{j1}) \end{aligned}$$

which implies that $x_{i2} \geq x_{j1} - x_{j2}$ and thus proves our claim. □

Proof of Theorem 5.1 Using Expression 8, one can see that ϕ_i for $i = 1, 2, S$ is maximized when $v(N) = v_{12S}$ is maximized, which happens when the pooled-inventory level for the 2-retailer coalition is set at the supply chain optimum level. □

Proof of Proposition 5.1 Employing the no-pooling strategy is one possible inventory management policy available to the coalition of two retailers and the supplier and therefore $v_{12S} \geq v_1 + v_2 + v_S$. The proposition follows from Expression 8. □

Proof of Proposition 5.2 By using Expression 8 we can easily verify that the allocations add up to v_{12S}, the value of the grand coalition. In addition, by Theorem 5.1 and Proposition 5.1, the second condition on the definition of core is satisfied. □

Proof of Theorem 6.1 In terms of S_1^{AP} and S_2^{AP} Expression 9 is:

$$\frac{\phi_1 + \phi_2}{p_M} \quad \geq \quad \frac{S_1^{AP} + S_2^{AP}}{2} \tag{13}$$

An equivalent expression to (13) is:

$$\begin{aligned} & \frac{S_1^{BP} + S_2^{BP}}{3} \left(1 - \frac{2p}{p_M}\right) + \frac{2c}{3p_M} \left[x_1^{BP} + x_2^{BP} - x_1^{AP} - x_2^{AP}\right] \\ & + \tfrac{2h}{3p_M} \left(H_1^{BP} + H_2^{BP} - H_1^{AP} - H_2^{AP}\right) \quad \geq \frac{S_1^{AP} + S_2^{AP}}{3} \left[1 - \frac{2p}{p_M}\right] \end{aligned} \tag{14}$$

Change in expected supplier profit exceeding average change in total expected retailer profit is represented as

$$\Delta\text{E[supplier profit]} \geq \frac{\Delta\text{E[total retailer profit]}}{2} \tag{15}$$

Using the definition of E[profit], we can rewrite inequality 15 as follows

$$\begin{aligned} &2p(S_1^{AP}+S_2^{AP}-S_1^{BP}+S_2^{BP})-2c(x_1^{AP}+x_2^{AP}-x_1^{BP}-x_2^{BP}) \\ &-2h(H_1^{AP}+H_2^{AP}-H_1^{BP}-H_2^{BP}) \quad \geq p_M(S_1^{AP}+S_2^{AP}-S_1^{BP}+S_2^{BP}) \end{aligned} \tag{16}$$

Algebraic manipulation reveals that inequality 16 is equivalent to Expression 13, which proves the claim. □

Proof of Proposition 6.1 In the proof of Theorem 6.1 we have established the equivalency of $\frac{\phi_1+\phi_2}{p_M} \geq S_1^{AP}+S_2^{AP}$ to Expression 16. Now rewriting Expression 16 using the Λ and Δ_i notation, we obtain

$$\frac{\phi_1+\phi_2}{p_M} \quad \geq \quad S_1^{AP}+S_2^{AP} \quad \Leftrightarrow \quad 2\Lambda \quad \geq \quad (p_M-2p)(\Delta_1+\Delta_2)$$

Similarly, we can write the following equivalent conditions.

$$\begin{aligned} \frac{\phi_1}{p_M} \quad &\geq \quad S_1^{AP} \quad \Leftrightarrow \quad \Lambda \quad \geq \quad (2p_M-p)\Delta_1-(p_M+p)\Delta_2 \\ \frac{\phi_2}{p_M} \quad &\geq \quad S_2^{AP} \quad \Leftrightarrow \quad \Lambda \quad \geq \quad (2p_M-p)\Delta_2-(p_M+p)\Delta_1 \end{aligned}$$

Without loss of generality, assume $\Delta_1 \geq \Delta_2$. The proposition states $2\Lambda \geq (p_M-2p)(\Delta_1+\Delta_2)$. This inequality along with $\Delta_1 \geq \Delta_2$ implies $\Lambda \geq (2p_M-p)\Delta_2-(p_M+p)\Delta_1$ which proves the result. □

Proof of Theorem 7.1 Let π_c^{AP} be the expected supply chain profit after pooling and π_S^{BP} be expected supplier profit before pooling. Rewriting Expression 7, the Shapley value allocation to retailer i is

$$\phi_i = \tfrac{2}{3}S_i^{BP}+\tfrac{1}{3}(\pi_c^{AP}-\pi_S^{BP}-p_M S_j^{BP})$$

By definition, only the last two terms of the above equation depend on ρ_j. Let $x_j(\rho_j)$ be the before-pooling stocking level for retailer j as a function of the service level. Then, $x_j(\rho_j) = F_j^{-1}(\rho_j)$. Let $\Omega = \frac{\partial F_j^{-1}(\rho_j)}{\partial \rho_j}$. Then,

$$\frac{\partial \phi_i(\rho_j)}{\partial \rho_j} \quad = \quad -\frac{\Omega}{3}(-c-h\rho_j+(p+p_M)(1-\rho_j))$$

Due to the assumptions we made on $F(\cdot)$, Ω is always positive. When $\rho_j < \frac{p+p_M-c}{p+p_M+h}$, then $\frac{\partial \phi_i(\rho_j)}{\partial \rho_j}$ is negative which means the function is decreasing and when $\rho_j > \frac{p+p_M-c}{p+p_M+h}$, the derivative is positive, which means the function is increasing. Therefore, the function is unimodal and $\rho_j^* = \frac{p+p_M-c}{p+p_M+h}$, is the global minimizer. □

Proof of Proposition 7.1 The Shapley value allocation to retailer i as a function of the service level of retailer j is

$$\begin{aligned}
\phi_i(\rho_j) &= \frac{2}{3}S_i^{BP} + \frac{1}{3}(\pi_c^{AP} - \pi_S^{BP} - p_M S_j^{BP}) \\
&= \frac{2}{3}S_i^{BP} + \frac{1}{3}\left(\pi_c^{AP} - \left(p(S_i^{BP} + S_j^{BP}) - h(H_i^{BP} + H_j^{BP}) - c(x_i + x_j)\right) - p_M S_j^{BP}\right) \\
&= K - \frac{1}{3}\left[(p + p_M - c)F_j^{-1}(\rho_j) - (p + p_M + h)\int_0^{F_j^{-1}(\rho_j)} F_j(x)dx\right]
\end{aligned}$$

where the term K represents the part of the $\phi_i(\rho_j)$ expression that does not depend on ρ_j and K is a function of ρ_i, p, p_M, h, and c. We can find the limit of the term in the parenthesis when $F_j(\cdot)$ has infinite support as follows:

$$\begin{aligned}
&\lim_{\rho_j \to 1}\left[(p + p_M - c)F_j^{-1}(\rho_j) - (p + p_M + h)\int_0^{F_j^{-1}(\rho_j)} F_j(x)dx\right] \\
&= \lim_{\rho_j \to 1}\left[(p + p_M + h)\int_0^{F_j^{-1}(\rho_j)} (1 - F_j(x))\, dx - (h + c)F_j^{-1}(\rho_j)\right] \\
&= (p + p_M + h)E[x] - (h + c)\lim_{\rho_j \to 1} F_j^{-1}(\rho_j) \\
&= -\infty
\end{aligned}$$

This implies $\lim_{\rho_j \to 1} \phi_i(\rho_j) = \infty$. □

Proof of Theorem 7.2 Let $\Omega_i = \frac{\partial F_i^{-1}(\rho_i)}{\partial \rho_i}$. Then

$$\frac{\partial \phi_S(\rho_i, \rho_j)}{\partial \rho_i} = \frac{\Omega_i}{3}(2(p - c) - p_M - (2(p + h) - p_M)\rho_i)$$

Since $F_i(\cdot)$ is a cumulative distribution function $\Omega_i > 0$ for $i \in \{1, 2\}$. It is sufficient to consider the following three cases.

- Case 1: $2(p - c) - p_M \geq 0$ and $2(p + h) - p_M \geq 0$
  In this case both $\frac{\partial \phi_S(\rho_1, \rho_2)}{\partial \rho_1}$ and $\frac{\partial \phi_S(\rho_1, \rho_2)}{\partial \rho_2}$ are positive over the interval $\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$ and negative over the interval $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}, \infty\right)$. Therefore both $\phi_S(\rho_1)$ and $\phi_S(\rho_2)$ are increasing over the interval $\left(0, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$ and decreasing over the interval $\left(\frac{2(p-c)-p_M}{2(p+h)-p_M}, \infty\right)$, which shows $\phi_S()$ is unimodal in both ρ_1 and ρ_2. For this region, the global maximum is at $(\rho_1, \rho_2) = \left(\frac{2(p-c)-p_M}{2(p+h)-p_M}, \frac{2(p-c)-p_M}{2(p+h)-p_M}\right)$.
- Case 2: $2(p - c) - p_M < 0$ and $2(p + h) - p_M \geq 0$
  In this region, for $\rho_1 \geq 0$ $\frac{\partial \phi_S(\rho_1, \rho_2)}{\partial \rho_1}$ is negative meaning $\phi_S(\rho_1)$ is decreasing. The same argument is true for $\frac{\partial \phi_S(\rho_1, \rho_2)}{\partial \rho_2}$ and $\phi_S(\rho_2)$. Therefore in this region $(\rho_1, \rho_2) = (0, 0)$ is the global maximum.

- Case 3: $2(p-c)-p_M<0$ and $2(p+h)-c_M<0$

  In this region $\frac{2(p-c)-p_M}{2(p+h)-p_M}\geq 1$ and beyond the meaningful service level region $[0,1)$. For $\rho_1\in[0,1)$ $\frac{\partial\phi_S(\rho_1,\rho_2)}{\partial\rho_1}$ is negative so $\phi_S(\rho_1)$ is decreasing. The same argument is true for $\frac{\partial\phi_S(\rho_1,\rho_2)}{\partial\rho_2}$ and $\phi_S(\rho_2)$. Therefore in this region $(\rho_1,\rho_2)=(0,0)$ is the global maximum.

□

Proof of Theorem 7.3 Let ϕ_i' and ϕ_j' denote the profit allocations to retailer i and j after the supplier maximizes her profit allocation. Define the following notation:

$$\begin{aligned}
\alpha &= \frac{2(p-c)-p_M}{2(p+h)-p_M}\\
\beta &= \frac{p-c}{p+h}\\
\nu &= F_i^{-1}(\alpha)\\
\eta &= F_i^{-1}(\beta)\\
\varepsilon &= F_j^{-1}(\alpha)\\
\gamma &= F_j^{-1}(\beta)
\end{aligned}$$

That the profit allocation to retailer i after the supplier maximizes her profit allocation is greater than or equal to retailer i's allocation under the base case, that is $\phi_i'\geq\phi_i$, is equivalent to

$$\begin{aligned}
&3p_M\left(\nu-\eta+\int_\nu^\eta F_i(x)\,dx\right)\\
&\quad\geq(-p-p_M+c)(\gamma-\varepsilon+\eta-\nu)+(p_M+p+h)\left[\int_\varepsilon^\gamma F_j(x)dx+\int_\nu^\eta F_i(x)\,dx\right]
\end{aligned}$$

and similarly $\phi_j'\geq\phi_j$ is equivalent to

$$\begin{aligned}
&3p_M\left(\varepsilon-\gamma+\int_\varepsilon^\gamma F_j(x)\,dx\right)\\
&\quad\geq(-p-p_M+c)(\gamma-\varepsilon+\eta-\nu)+(p_M+p+h)\left[\int_\varepsilon^\gamma F_j(x)\,dx+\int_\nu^\eta F_i(x)\,dx\right]
\end{aligned}$$

The total after-pooling profit of the supply chain does not increase when the supplier maximizes her own after-pooling profit allocation. Then both of the inequalities cannot hold at the same time. Either neither of the equalities will hold or only one of them will hold. Therefore we need to compare $\nu-\eta+\int_\nu^\eta F_i(x)dx=\int_\nu^\eta(-1+F_i(x))\,dx$ and $\varepsilon-\gamma+\int_\varepsilon^\gamma F_i(x)dx=\int_\varepsilon^\gamma(-1+F_j(x))\,dx$ to find which retailer's profit allocation increases, if any. Since $D_i\geq_{disp}D_j$, we have $\eta-\nu\geq\gamma-\varepsilon$.

Since $D_i \geq_{disp} D_j$, we have

$$F_i^{-1}(1-y) - F_j^{-1}(1-y) \geq F_i^{-1}(1-x) - F_j^{-1}(1-x) \tag{17}$$

for $y \leq x$ and $y, x \in [1-\beta,\ 1-\alpha]$. Expression 17 implies that $1 - F_i(\nu+\delta) \geq 1 - F_j(\varepsilon+\delta)$ for $\delta \in [0, \gamma-\varepsilon]$ and that $\eta - \nu \geq \gamma - \varepsilon$. Then $\int_\nu^\eta (-1+F_i(x))\ dx \leq \int_\varepsilon^\gamma (-1+F_j(x))\ dx$, which concludes the proof. $\square$