# Robust Parameter Design of Multiple Target Systems

V. Roshan Joseph and C.F. Jeff Wu

Department of Statistics

University of Michigan

Ann Arbor, MI 48109 -1285

**Abstract**

In multiple target systems, the signal factor is varied based on the signal-response relationship to achieve different targets for the response specified by the customer. Robust parameter design aims at making the signal-response relationship insensitive to the noise variation by choosing appropriate levels for the control factors. Taguchi's dynamic signal-to-noise ratio has several limitations for the optimization in multiple target systems. We give a theoretical formulation of the problem and develop a practical approach for optimization that overcomes these limitations. The methodology is illustrated using a temperature controller example.

KEY WORDS: Signal-response system, Signal-to-noise ratio, Experiments, Optimization.

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# 1. INTRODUCTION

Robust parameter design aims at finding the levels of control factors that will make the effect of noise factors on the performance of the system as small as possible. When the customer puts forth different requirements on the system performance, all of the control factors cannot be kept at a constant level. In such cases a factor known as signal factor is selected from the set of control factors and is changed continuously depending on the customer intent to meet his requirements (see Figure (1)). For examples, different hole diameters on a job are obtained by using different drill bit sizes in a drilling operation; different line widths for a circuitry in a printed circuit board are produced by giving different line widths in the art work used in the image transfer process. The user of the system has to specify a value for the signal factor to achieve a specific customer intent and hence he/she needs to know the relationship between the signal factor and the output response. The system will work well if this relationship is robust to the noise factors. The approach of making the signal-response relationship insensitive to noise was introduced by Taguchi (1987) and is known as *dynamic parameter design* or *parameter design in signal-response systems.* The recent developments in this area can be found in Miller and Wu (1996), Wu and Hamada (2000) and Wu and Wu (2000). The increasing competition in the market and changing demands from the customers have forced the manufacturer to design and produce products with varied functional requirements. It became a necessity not only to meet the current requirements but also to develop the process anticipating the future requirements. Dynamic parameter design has emerged as an important engineering-statistical tool in this technology development process. Many interesting case studies from industries can be found in Taguchi, Chowdhury, and Taguchi (2000).

Taguchi (1987, 1993) uses a linear relationship between signal ($M$) and response ($Y$) given by

$$Y = \beta M + \epsilon, \tag{1}$$

where $\epsilon$ has mean 0 and variance $\sigma^2$. The $\beta$ and $\sigma$ are evaluated for different control factor settings and a setting is selected that will optimize a performance measure of variation from

the linear relationship. He uses the (dynamic) signal-to-noise ($SN$) ratio given by

$$SN = \log \beta^2/\sigma^2 \tag{2}$$

as the performance measure. Miller and Wu (1996) confirms the suitability of this measure for optimization in *measurement systems* but comments on its inadequacy for dealing with *multiple target systems.* They propose a response function modeling approach for the analysis. Lunani, Nair, and Wasserman (1997) advocates the use of a generalized signal-to-noise ratio

$$GSN = \log \beta^\gamma/\sigma^2 \tag{3}$$

as the performance measure and gives some graphical methods for obtaining $\gamma$ from the data.

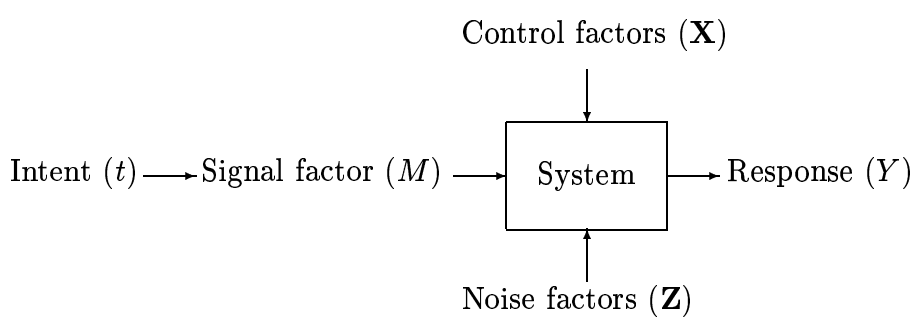


Figure 1: Multiple Target System

In an engineered system, the actual relationship need not be linear. Taguchi (1990) treats the nonlinearity as undesirable, considering it a "distortion in engineering fields". The $\sigma$ calculated from model (1) is a combined measure of variation due to noises and departure from linearity. There are many situations in which a system can work well with a nonlinear relationship as long as the relationship is well understood and is easily implementable in the system. Hence forcing a system to behave with a linear signal-response relationship can result in a sub-optimal design of the system. But modeling a non-linear relationship in an experiment can be expensive as the data needs to be collected for many levels of signal factor for every control factor settings. It will be shown that the optimization procedure is

very complex with a nonlinear relationship. We will develop some methods to address the nonlinearity while keeping the optimization simple to perform.

In model (1) we see that as $M$ is reduced to zero, the mean of $Y$ is also reduced to zero. If $Y$ is allowed to take only non-negative values, then the variation in $Y$ will also reduce to zero. The statistical model in (1) does not capture this property, leading to inefficient estimates of $\beta$ and $\sigma$. We will stress on the importance of modeling variation in $Y$ as a function of signal factor as it is crucial in obtaining the right performance measure for robust design optimization.

The article is organized as follows. In Section 2 we give some typical examples to show that variance will usually be a function of the signal factor. In Section 3 we describe a general approach to the robust design optimization of multiple target systems. In Section 4 we develop two practical approaches to modeling. In Section 5 the underlying statistical model for signal-to-noise ratio is identified. The proposed methodology is illustrated with an example in Section 6. Conclusions are given in Section 7.

## 2. VARIANCE FUNCTION

A signal factor can be used to cause large changes in the mean of a response. Changes in the variation is bound to happen when the mean is varied a lot and hence the variation will be a function of the signal factor.

*A temperature controller example*: Phadke (1989) describes the robust design of a temperature controller. The resistance at which the heater will be turned on is given by

$$R_{T-on} = \frac{R_3 R_2 (R_4 + R_1 E_0/E_Z)}{R_1 (R_4 - R_2 (E_0/E_Z - 1))}. \tag{4}$$

More details about the functioning of this circuit can be found in McCaskey and Tsui (1997). The resistance $R_3$ is the signal factor and variation in the other factors within their manufacturing tolerance is the noise to the system. Here $Var(R_{T-on}) \propto R_3^2$. An approximate statistical model for this system is

$$R_{T-on} = \beta R_3 + \epsilon,$$

where $E(\epsilon) = 0$ and $Var(\epsilon) = \sigma^2 R_3^2$. The $\beta$ and $\sigma^2$ are functions of the nominal values of $R_1, R_2, R_4, E_0$, and $E_z$.

*An electro plating example*: Consider an electro plating process used to deposit metal onto another metallic substrate. The plating time can be varied to achieve different thicknesses of the plated metal. By Faraday's law the plating thickness (mean) is proportional to the plating time. Let $V(\tau)$ represents the variability in thickness during time $\tau$. Divide the interval $(0, \tau)$ into $n$ equal parts of length $\tau/n$ as $(0, \tau/n), ..., ((n-1)\tau/n, \tau)$. The amount of metal deposited in each of these intervals can be assumed to be independently and identically distributed. Then $V(\tau) = nV(\tau/n)$ for all $n = 1, 2, 3, ....$ This is true if and only if $V(\tau) \propto \tau$. Hence the statistical model for this system can be written as,

$$Y = \beta\tau + \epsilon,$$

where $E(\epsilon) = 0$ and $Var(\epsilon) = \sigma^2\tau$. The variability $V(\tau)$ is caused by the fluctuations in some of the noise factors during time $\tau$. If we absorb the other noise factors also into the variance term, the variation can be modeled as $\sigma_1^2\tau + \sigma_2^2\tau^2$, which can be approximated as $\overline{\sigma}^2\tau^\alpha$. The $\alpha$ is a constant between 1 and 2, depending on the magnitudes of $\sigma_1^2$ and $\sigma_2^2$ in the experimental region.

## 3. GENERAL FORMULATION

Two types of systems are usually encountered in dynamic parameter design problems: measurement systems and multiple target systems. In a measurement system, based on the measured value the true value is to be estimated with minimum variability. This is a problem of inverse regression. Using the concept of Fieller interval, Miller and Wu (1996) proved that the $SN$ ratio in (2) is an appropriate performance measure for optimization. In a multiple target system, the signal factor is to be adjusted so as to achieve several targets for the response. The objective is to design the system so that the signal-response relationship is robust to noise factors. In this section, we describe a general approach to the robust design of multiple target systems.

Let $E_{\mathbf{Z}}(Y) = f(\mathbf{X}, M)$ and $Var_{\mathbf{Z}}(Y) = V(\mathbf{X}, M)$, where $\mathbf{X}$ is the set of control factors. The expectation and variance are taken over the distribution of the noise factors $\mathbf{Z}$. The expected quality loss assuming a quadratic loss function is proportional to

$$L(\mathbf{X}, M, t) = (f(\mathbf{X}, M) - t)^2 + V(\mathbf{X}, M),$$

where $t$ is the customer intent. Given $t$, the signal factor can be adjusted to minimize the loss. Cost considerations or system limitations may require the value of $M$ to be between $M_L$ and $M_H$. Let

$$\min_{M \in (M_L, M_H)} L(\mathbf{X}, M, t) = L(\mathbf{X}, M^*, t). \tag{5}$$

Let $t$ be any value between $a$ and $b$ and $W(t)$ is the probability distribution function. Because $\mathbf{X}$ cannot be varied with $t$, the expected loss at $M = M^*$ with respect to the customer intent can be used as a measure to evaluate the control factor settings. Thus the performance measure for a given $\mathbf{X}$ is

$$PM = \int_a^b L(\mathbf{X}, M^*, t) dW(t). \tag{6}$$

Our objective is to find an $\mathbf{X} \in D$ minimizing $PM$, where $D$ is the feasible region for $\mathbf{X}$. The minimization of the loss in (5) need not result in a value of $M$ to get the expected value of the response at target $t$. But in engineering applications an unbiased adjustment strategy makes more sense. Therefore we can simplify (6) by avoiding the minimization over $M$. Let $h(\mathbf{X}, t)$ be the solution of $M$ from $f(\mathbf{X}, M) = t$. A unique solution exists because the signal-response relationship is assumed to be a continuous monotonic function. Now the $PM$ can be viewed as the average variability in $Y$ after adjustment. This simplification is possible only if the evaluation of the mean and variance functions at $h(\mathbf{X}, t)$ is meaningful. The $h(\mathbf{X}, t)$ need not lie in $(M_L, M_H)$ for all $\mathbf{X}$. So we need to search for an $\mathbf{X}$ satisfying this requirement. Thus the optimization problem becomes

$$\min_{\mathbf{X} \in D} PM = \int_a^b V(\mathbf{X}, h(\mathbf{X}, t)) dW(t) \tag{7}$$

subject to

$$\max_{t \in (a,b)} h(\mathbf{X}, t) \leq M_H, \tag{8}$$

and

$$\min_{t \in (a,b)} h(\mathbf{X}, t) \geq M_L. \tag{9}$$

Because the objective function and constraints are not given explicitly as functions of $\mathbf{X}$, the optimization is difficult to perform. Therefore it is required to recast the formulation into a more tractable form. Also the approach has some practical limitations because in general the mean and variance functions are not known and are estimated from the experimental data. Discussion on the choice of experimental plans for signal-response systems can be found in Wu and Hamada (2000, Section 11.6). If $f(\mathbf{X}, M)$ is a nonlinear function in $M$, then data needs to be collected at many levels of $M$. Because in most experiments the number of runs is a multiple of the signal factor levels, the experiment becomes expensive to conduct. From the cost point of view it is beneficial to have few signal factor levels and therefore few parameters in the mean and variance function models. We will show that this can be done and still achieve the stated objectives.

# 4. MODELING

In this section we develop two different approaches to modeling. In *Mean-Variance modeling* we will separately model mean and variance functions. This modeling is ideal when the noise levels are randomly chosen for every observation. In *Response modeling* the response is directly modeled in terms of the control, noise, and signal factors. This approach is suitable when the noise factors have fixed levels in an experiment.

## 4.1 Mean-Variance Modeling

We explain the ideas behind Mean-Variance modeling by choosing some special functional forms for both mean and variance. Consider an additive noise model $Y = f(\mathbf{X}, M) + \epsilon$, where $E(\epsilon) = 0$ and $Var(\epsilon) = V(\mathbf{X}, M)$. Assume that the response and signal factor are nonnegative variables and $f(\mathbf{X}, 0) = 0$. Also assume that the mean and variance are increasing functions of $M$, perhaps after a suitable transformation of the signal factor. Based on the discussions in Section 2,

$$V(\mathbf{X}, M) = \sigma^2(\mathbf{X}) M^{\alpha} \tag{10}$$

is a reasonably good approximation for the variance function.

We will first consider a special form of $f(\mathbf{X}, M)$ and develop the optimization procedure. As will be seen later, this form is helpful to deal with a more general case using lack of fit. Let

$$f(\mathbf{X}, M) = f_\theta(\beta(\mathbf{X})M), \tag{11}$$

where $f_\theta$ is a known function up to a set of parameters $\theta$. Because the mean and variance functions are meaningful only for nonnegative values of $M$, we must assume that $f_\theta^{-1}(a) \geq 0$ to ensure unbiased adjustments for $t \in (a, b)$. Typically $f_\theta(0) = 0$ and hence this will be automatically satisfied. From (10) and (11), the $PM$ in (7) becomes

$$PM = \int_a^b \sigma^2(\mathbf{X}) \left( \frac{f_\theta^{-1}(t)}{\beta(\mathbf{X})} \right)^\alpha dW(t) = \frac{\sigma^2(\mathbf{X})}{\beta^\alpha(\mathbf{X})} \int_a^b (f_\theta^{-1}(t))^\alpha dW(t).$$

With the selected mean function in (11), the $\mathbf{X}$ and $t$ are *separated* in the $PM$, thus simplifying the procedure. Define the performance measure as

$$\eta(\mathbf{X}) = \log \frac{\beta^\alpha(\mathbf{X})}{\sigma^2(\mathbf{X})} \tag{12}$$

to be consistent with the form of $SN$ ratio in (2). Note that we no longer need to specify the probability distribution of the customer intent to get the performance measure. Thus the optimization problem is reduced to

$$\max_{\mathbf{X} \in D} \eta(\mathbf{X})$$

$$\text{subject to } \beta(\mathbf{X}) \geq \beta_L, \text{ where}$$

$$\beta_L = f_\theta^{-1}(b)/M_H, \tag{13}$$

which is obtained from the constraint in (8). The constraint (9) is trivially satisfied for $M_L = 0$. The above problem can be solved using a mathematical programming algorithm. See, for example, Luenberger (1989). The two-step optimization procedure is found to work well in many practical situations. The two-step optimization will succeed if there exists an adjustment parameter. An *adjustment parameter* in this case refers to a variable that has an effect on $\beta$ but not on $\eta$. The two-step optimization procedure in our formulation can be stated as follows.

1. Find $\mathbf{X} \in D$ to maximize $\eta(\mathbf{X})$.

2. Use one or more adjustment parameters to adjust $\beta$ to the desired range.

$$(14)$$

Note that in the formulation $\beta$ is not bounded above. A very high sensitivity can become undesirable if there is error in the signal factor settings. However if the error is also treated as a noise factor, then such a case will not arise.

Now we will consider the issue of estimation. Assume that the data is obtained using a cross array design which is a product of the control array, noise array, and signal factor levels. Let $y_{ijk}$ denote the response value at run $i$, signal level $j$, and noise level $k$. Then

$$y_{ijk} = f(\mathbf{X}_i, M_j) + \epsilon_{ijk}, \tag{15}$$

where $\epsilon_{ijk}$'s are independently distributed as $N(0, \sigma_i^2 M_j^\alpha)$. Suppose we do not know the true functional form of the mean. Then $\overline{y}_{ij.}$ is an obvious estimate of $f(\mathbf{X}_i, M_j)$. Hence $\sigma_i^2 M_j^\alpha$ can be estimated from

$$s_{ij}^2 = \frac{1}{K-1} \sum_{k=1}^{K} (y_{ijk} - \overline{y}_{ij.})^2. \tag{16}$$

The estimation of $\sigma_i^2$ and $\alpha$ can be done using a log-linear model,

$$\log \sigma_{ij}^2 = \log \sigma_i^2 + \alpha \log M_j, \tag{17}$$

where $(K-1)s_{ij}^2/\sigma_{ij}^2 \sim \chi_{K-1}^2$. The fitting can be done using gamma GLM with a log-link or using the quasi-likelihood estimation (McCullagh and Nelder, 1989). Thus we can compute the noise variability without actually knowing the exact mean function. To find the variability after adjustment, we have to know the value of $M$ to get the response at $t$ on average. Instead of using $f(\mathbf{X}, M)$, an approximate value of $M$ can be obtained by fitting $f_\theta(\beta(\mathbf{X})M)$ to the data. The model $Y \sim f_\theta(\beta M)$ is to be fitted using weighted least squares with weights $(\sigma^2 M^\alpha)^{-1}$. Note that

$$\sum_i \sum_j \sum_k (y_{ijk} - f_\theta(\beta_i M_j))^2/\sigma_i^2 M_j^\alpha = K \sum_i LOF_i/\sigma_i^2 + (K-1) \sum_i \sum_j s_{ij}^2/\sigma_i^2 M_j^\alpha, \tag{18}$$

where the lack of fit sum of squares is

$$LOF_i = \sum_j (\overline{y}_{ij.} - f_\theta(\beta_i M_j))^2 / M_j^\alpha.$$

In Taguchi's approach, the $\sigma^2$ in model (1) includes both the lack of fit and the noise variation. Here we separate these two components and try to minimize only the noise variation. The cause for lack of fit is not random and therefore the robustness should not be sacrificed to reduce the lack of fit. The minimization of $LOF$ is not important because in most cases it can be eliminated at the optimal setting by fitting a more elaborate signal-response relationship. The signal-response relationship is usually restricted to be a monotonic function. Therefore, if the lack of fit cannot be completely eliminated with elaborate modeling, the cause behind it needs to be identified and removed from the system. This approach helps us to avoid the need of fitting elaborate signal-response models to each run, thereby making the experiment less expensive. In situations where specific functional forms such as linearity is preferred in a system, the $LOF$ can be separately modeled and minimized. The decomposition like (18) was used in Miller and Wu (1996) in the analysis of an injection molding experiment. Our approach is more general as the separation of lack of fit from the residual variation is introduced as a strategy to deal with nonlinear signal-response systems.

*Summary*: The steps in modeling and optimization are summarized as follows.

1. Estimate $\sigma_i^2$ and $\alpha$ by fitting a log-linear model in (17) using a gamma GLM, where $s_{ij}^2$ is estimated from (16).

2. Estimate $\beta_i$ and $\theta$ from $y_{ijk} \sim f_\theta(\beta_i M_j)$ with weights $(\hat{\sigma}_i^2 M_j^{\hat{\alpha}})^{-1}$.

3. Fit $\log \hat{\beta}^{\hat{\alpha}}/\hat{\sigma}^2 \sim \mathbf{X}$ and $\log \hat{\beta} \sim \mathbf{X}$.

4. Use a mathematical programming algorithm or the two-step optimization procedure in (14) to find the optimal $\mathbf{X}$.

5. Collect more data at the optimal $\mathbf{X}$ for different values of the signal factor and fit a more elaborate signal-response model.

The models used in steps 1 and 2 work only under assumptions as previously discussed. More general models could be used, but it will make the computations of the performance measure and the optimization more complicated.

### 4.2 Response Modeling

When the noise factors have fixed settings in an experiment, the $\epsilon_{ijk}$'s in (15) are not independent and therefore an approach that takes into account of this structure is more appropriate. See the discussions in Bérubé and Nair (1998). Let $\mathbf{N}$ be the set of observable noise factors. Then for a given $\mathbf{N}$, let $Y = f(\mathbf{X}, \mathbf{N}, M) + \epsilon$, where $E(\epsilon) = 0$, $Var(\epsilon) = \phi(\mathbf{X}, \mathbf{N}, M)$ and $\epsilon$ is the random error caused by the unobservable noise factors. As an approximation to the true signal-response relationship, consider a polynomial model

$$f(\mathbf{X}, \mathbf{N}, M) = \beta_0(\mathbf{X}, \mathbf{N}) + \beta_1(\mathbf{X}, \mathbf{N})M + \cdots + \beta_r(\mathbf{X}, \mathbf{N})M^r. \tag{19}$$

In the response function modeling suggested by Miller and Wu (1996), the parameters in (19) are estimated for each control and noise combinations and then they are modeled with respect to $\mathbf{X}$ and $\mathbf{N}$. An alternative approach is to fit (19) directly from the data. See Miller (1993) and Tsui (1999). An important issue not considered in the literature is that the signal-response relationship should be monotonic. Polynomial models do not posses the monotonicity property. Therefore the estimation procedure should ensure that the fitted response is monotonic in $M$ in the desired range. Most of the research in monotonic regression is on non-parametric methods. See Robertson, Wright, and Dykstra (1988). The estimation problem can be avoided if the regression function is chosen from a monotone class of functions.

As in the Section 4.1, choose an appropriate model for the variance term and estimate from replicates. Then (19) can be fitted with weights $1/\hat{\phi}(\mathbf{X}, \mathbf{N}, M)$ using monotonic regression. It is not necessary to reestimate $\phi(\mathbf{X}, \mathbf{N}, M)$ from the residuals if we decided to throw out the lack of fit component. The mean and variance functions are then obtained using the conditional expectation and variance formulas. This requires explicit knowledge about the distribution of noise factors and may need to use Monte Carlo methods. Most often the $PM$

in (7) has to be obtained using numerical integration for different control factor settings. The $PM$ can then be modeled with respect to $\mathbf{X}$ and optimized.

For a special case with $f(\mathbf{X}, \mathbf{N}, M) = \beta_1(\mathbf{X}, \mathbf{N})M$ and $\phi(\mathbf{X}, \mathbf{N}, M) = \phi(\mathbf{X}, \mathbf{N})M^2$, the $PM$ in (7) can be explicitly obtained as

$$PM \equiv \frac{Var_{\mathbf{N}}\beta_1(\mathbf{X}, \mathbf{N}) + E_{\mathbf{N}}\phi(\mathbf{X}, \mathbf{N})}{E^2_{\mathbf{N}}\beta_1(\mathbf{X}, \mathbf{N})}$$

which can be viewed as the reciprocal of the signal-to-noise ratio. Consider the case of a single noise factor with mean 0 and variance $\sigma_1^2$. If we further assume $\beta_1(\mathbf{X}, \mathbf{N}) = \beta_{10}(\mathbf{X}) + \beta_{11}(\mathbf{X})N_1$ and $\phi(\mathbf{X}, \mathbf{N}) = \phi_0(\mathbf{X}) + \phi_1(\mathbf{X})N_1$, then the optimization problem becomes

$$\max_{\mathbf{X} \in D} \frac{\beta_{10}^2(\mathbf{X})}{\beta_{11}^2(\mathbf{X})\sigma_1^2 + \phi_0(\mathbf{X})}$$

$$\text{subject to } \tfrac{b}{M_H} \leq \beta_{10}(\mathbf{X}) \leq \tfrac{a}{M_L}$$

which can be easily solved using a standard nonlinear programming algorithm.

Compared to Miller and Wu (1996) and Tsui (1999), we focus on minimizing the variation in the response after adjusting for the mean. This is important because the variance is a function of the signal factor and can change while adjusting the mean to a specified target. Tsui (1999) omitted the signal×noise interaction term from modeling which we consider to be an important ingredient in the signal-response models as it forms the basis of variance function.

### 4.3 Discussion

The performance of the $SN$ ratio in (2) is comparable to the $\eta$ in (12) when $\alpha = 2$ and with a linear signal-response relationship. Because of the importance of $SN$ ratio, a more rigorous treatment will be given in the next section. For estimating the $SN$ ratio, Taguchi (1993) uses an unbiased estimate of $\beta^2$. Because the performance measure is used only for comparing different control factor settings, the bias is not of much concern as long as it does not change greatly with $\mathbf{X}$. In the Appendix we show that the bias in maximum likelihood estimate can be neglected.

Transformations on both signal factor and response should be used to enhance the modeling described in the previous subsections. There are several experiments reported that use transformations based on engineering knowledge to linearize the signal-response relationship. See, for example, Fowlkes and Creveling (1995). The robust setting obtained using the transformed response can be considered to be an approximation to the robust setting for the original response under the unbiased strategy. It is also possible to introduce unknown parameters in the transformations. In such a case the estimation of the parameters should be done using the iteratively re-weighted least squares method. See McCullagh and Nelder (1989) and Engle and Huele (1996).

Although the performance measure in (12) resembles the generalized signal-to-noise ratio in (3), they are entirely different. The $\gamma$ in (3) is obtained by assuming $\sigma^2 \propto \beta^\gamma$ in model (1) and hence the value of $\gamma$ will be different from $\alpha$, depending on how the control factors affect $\beta$ and $\sigma$. As we have seen, the minimization of variability demands a performance measure depending on how the signal factor affect the mean and variance and not on the overall slope-variance relationship exhibited in the data. Because the static characteristics (i.e. response with a single target value) is a special case of dynamic characteristics, similar comments apply to the static parameter design optimization. As will be shown in the next section, the best non-informative choice of $\alpha$ is 2 and therefore the static $SN$ ratio analysis is justifiable in many linear systems.

## 5. SIGNAL-TO-NOISE RATIO

In this section we will identify the underlying statistical model for the signal-to-noise ratio analysis. The explanation of the signal-to-noise ratio given here is different from those in the literature. Taguchi (1993) assumes that when a scaling factor (same as the adjustment parameter) is used to adjust the $\beta$ to its ideal value $\beta_I$, the variance will change to $\sigma^2(\beta_I/\beta)^2$ and therefore uses the $SN$ ratio in (2) for optimization. Phadke and Dehnad (1988) extends the same idea in their derivation. Leon, Shoemaker, and Kacker (1987) justifies the $SN$ ratio as a performance measure independent of adjustment (PerMIA) under a model with $Var(Y) \propto \beta^2$. See also Lunani, Nair, and Wasserman (1997) and Wu and Hamada (2000).

We will show that it is not required to assume the existence of a scaling factor and will derive a version of the signal-to-noise ratio under some assumptions.

Assume that $Y$ and $M$ are nonnegative variables and the signal-response relationship passes through the origin. We will also assume that the response is not the end result of an additive process to avoid situations like the electro-plating example given in Section 2. Let $\mathbf{Z}$ be the set of noise factors. Then the relationship between the response and other factors can be written as

$$Y = f(\mathbf{X}, \mathbf{Z}, M).$$

Using Taylor's theorem and series expansion,

$$Y = M \left[\frac{\partial f}{\partial M}\right]_{M=\overline{M}} = M \left[\frac{\partial f}{\partial M}\right]_{M=M_0} + M(\overline{M} - M_0) \left[\frac{\partial^2 f}{\partial M^2}\right]_{M=M_0} + \cdots$$

where $0 \leq \overline{M} \leq M$ and $M_0$ is chosen so as to get a good linear fit through the origin in the region of interest. Using only the first term in the expansion, we get the following two approximate formulas

$$E_{\mathbf{Z}}(Y) = M\, E_{\mathbf{Z}} \left[\frac{\partial f}{\partial M}\right]_{M=M_0}, \tag{20}$$

$$Var_{\mathbf{Z}}(Y) = M^2\, Var_{\mathbf{Z}} \left[\frac{\partial f}{\partial M}\right]_{M=M_0}. \tag{21}$$

Thus an approximate statistical model under the stated assumptions is

$$Y = \beta M + \epsilon, \tag{22}$$

where $E(\epsilon) = 0$ and $Var(\epsilon) = \sigma^2 M^2$. Following the approach of Section 4.1, the performance measure to maximize can be derived as $\beta^2/\sigma^2$. Now $\beta^2/\sigma^2 = E^2(Y)/Var(Y)$, which can be viewed as a signal-to-noise ratio. The performance measure from model (1) is $\sigma^2$ and is not the $SN$ ratio. Thus the underlying statistical model of the signal-to-noise ratio is different from (1).

In the Mean-Variance modeling, using $f_\theta(\beta M)$ for the mean function will improve the approximation in (20), whereas using $\alpha$ instead of 2 and separating lack of fit from the residuals were aimed at improving the approximation in (21) and to relax some of the

assumptions. Also the dependence of variance with $M$ necessitates the use of weighted least squares estimation. Hence we can expect an overall improvement in the analysis by using the performance measure in (12) compared to the $SN$ ratio. We will substantiate this claim with an example in the next section.

## 6. AN EXAMPLE

We will illustrate the proposed methodology using the temperature controller example. For the purpose of illustration we will treat $R_2$ as the signal factor instead of $R_3$ so that we have a nonlinear signal-response relationship. We will assume no knowledge about the exact form of the transfer function in (4) for the analysis and then evaluate the performance of our approach based on the transfer function. The factors and their levels are given in Table. 1.

Suppose the customer requirement on $R_{T-on}$ is uniformly distributed in $(1, 5)$ and the maximum value for $M$ that can be used to achieve this range is 4. Select three levels for $M$ as 1, 2, and 3.5. Assume that the deviations in the factors from their nominal values are normally distributed with standard deviation 5% of their nominal values. Two levels of compounded noise factor is selected as

$N1$ (low $R_{T-on}$) : $1.05A/0.95B/1.05C/0.95D$

$N2$ (high $R_{T-on}$): $0.95A/1.05B/0.95C/1.05D$,

where $A, B, C$, and $D$ are the nominal values of the 4 factors. A $2^4$ design is used for the control factors. The response $R_{T-on}$ is generated from the transfer function in (4). For each run there are 6 values of $R_{T-on}$ corresponding to the 3 levels of the signal factor and 2 levels of the noise factor.

For comparison we will use both Mean-Variance modeling and Response modeling for analysis. The $R_{T-on}$ values are plotted against the signal factor for each control and noise combination and is given in Figure 2. We see that the mean and variance of $R_{T-on}$ increases with $M$. Assume the experimenter knows that the $R_{T-on}$ is zero when $M = 0$. Using gamma GLM, $\alpha$ is estimated from the log-linear model in (17) as 2.58. An approximate 95% confidence interval for $\alpha$ is (2.45, 2.72), which clearly shows a value more than 2 should be used in the analysis. Let $f_\theta(\beta M) = (\beta M)^\theta$. Now fit $R_{T-on} \sim (\beta M)^\theta$ using nonlinear

Table 1: Factors and levels

| Sl.No. | Factors | Notation | Levels | |
|---|---|---|---|---|
| | | | 1 | 2 |
| 1 | $R_1$ | $A$ | 2 | 4 |
| 2 | $R_3$ | $B$ | 2 | 4 |
| 3 | $R_4$ | $C$ | 15 | 25 |
| 4 | $E_0/E_z$ | $D$ | 1.5 | 2.5 |

regression with weights $(\hat{\sigma}^2 M^{\hat{\alpha}})^{-1}$. This gives the value of $\theta$ as 1.111. The half-normal plot of the effects of $\eta = \log \hat{\beta}^{\hat{\alpha}}/\hat{\sigma}^2$ is given in Figure 3a. It shows that the four main effects and the $CD$ interaction are significant. The optimum levels are $A_1B_2C_2D_1$. The lower bound for $\beta$ can be calculated from (13) as 1.06. The value of $\beta$ at the optimum combination (run no. 7) is 2.04 which satisfies the constraint and hence the adjustment step in (14) is not required. At the optimum setting an approximate range of $M$ to get $R_{T-on}$ in (1,5) is obtained as (0.49,2.08). So generate data by taking $M$ between 0.25 and 2.5 with an increment of 0.25. One model that fits this data well is

$$R_{T-on} = 2.24M + 0.0447M^2 + 0.00099M^3,$$

which can be used to design the adjustment system in the temperature controller.

For the Response modeling a quadratic polynomial with no intercept term is fitted. The error in the model is solely due to the lack of fit and therefore it is omitted from analysis. The performance measure in (7) is obtained through numerical integration and is denoted by $PM_R$. The half-normal plot of $\log PM_R$ given in Figure 3b shows that the four main effects are significant.

Now we will see the results of exact analysis based on the transfer function. Introduce $\lambda$ so that

$$ER_{T-on} = \lambda \frac{B(C + AD)}{A(C - (D-1)M)} M. \tag{23}$$

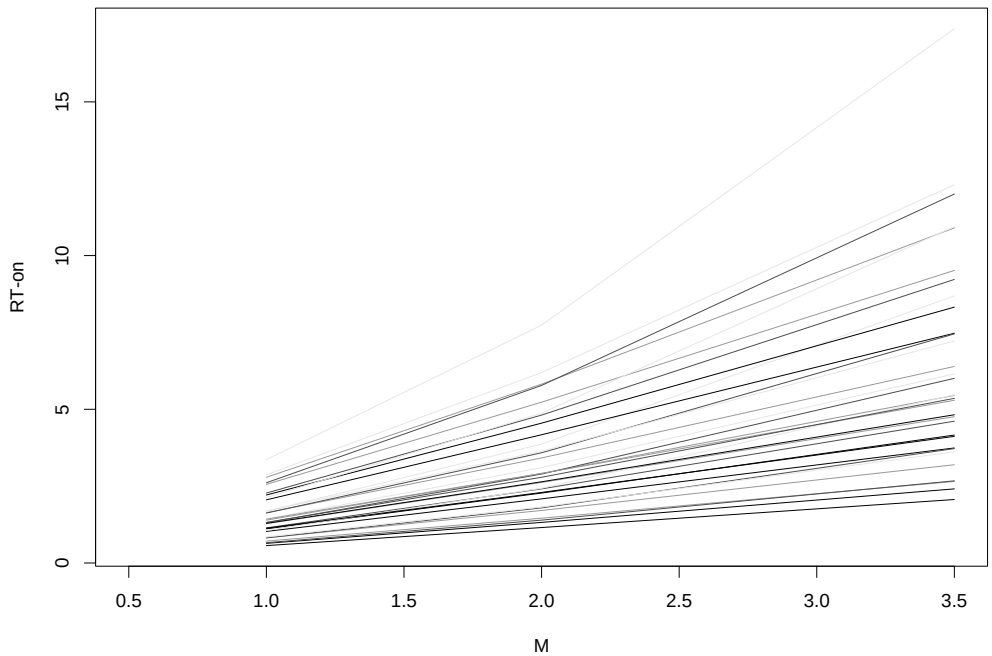


Figure 2: Plot of $R_{T-on}$ against $M$

To get $ER_{T-on} = t$, the value of $M$ should be

$$M = \frac{ACt}{(D-1)At + \lambda BC + \lambda ABD}. \tag{24}$$

Starting with $\lambda = 1$ and using simulations to compute $ER_{T-on}$, $M$ can be iteratively solved from (23) and (24). This value of $M$ is then used to compute $Var(R_{T-on})$ based on 10,000 simulations. The variance is computed for $t$ =1, 2, 3, 4, and 5 and is numerically integrated using a 5-point rule to get the $PM$ in (7). The half-normal plot of the effects of $\log PM$ given in Figure 3c shows that only the main effects are important. The optimum combination is $A_1B_2C_2D_1$ which is the same setting as obtained earlier.

We also analyze this problem using the $SN$ ratio in (2). For each run, $\beta$ and $\sigma$ are estimated based on model (1) using least squares and the $SN$ ratio is calculated. The $\eta$ values and $SN$ values are plotted against $\log(1/PM)$ and is given in Figure 4. As expected, $\eta$ is more correlated with the exact performance measure than $SN$, the $R^2$ values being 81.3% and 37.4% respectively. The half-normal plot of the effects of $SN$ is shown in Figure 3d. We

see that the $SN$ ratio analysis completely missed the most important effect of $B$ and barely identifies $A$ as significant. The analysis of $\eta$ and $\log PM_R$ did not detect the effects in the exact order of importance (compare Figures 3 a,b, and d) but succeeded in capturing the important effects.

## 7. CONCLUSIONS

In this paper we have formulated the robust parameter design of multiple target systems as a mathematical programming problem. We have defined the performance measure to evaluate a control factor setting as the average variability in the response after adjusting for the mean through the signal factor. We have explained two practical approaches for the estimation of the performance measure. The signal-response relationship in many multiple-target systems can be nonlinear. Some methods to address the nonlinearity are proposed. The usefulness of the approach is demonstrated using an example.

The statistical model underlying the signal-to-noise ratio analysis is shown to be (22), which is different from the widely used model in (1). It is believed that the signal-to-noise ratio is justified only when $Var(Y) \propto \beta^2$ which requires the existence of a scaling/adjustment factor. We have shown that this is not true. It is how the signal factor affects the mean and variance that leads to the signal-to-noise ratio or a related performance measure. Throughout the article we have emphasized on the importance of modeling variation in the response as a function of the signal factor to get the right performance measure for robust design optimization.

## ACKNOWLEDGMENTS

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## APPENDIX: BIAS IN ESTIMATION

Consider the signal-to-noise ratio model,

$$y_{ijkl} = \beta_i M_j + \epsilon_{ijkl}$$

where $y_{ijkl}$ is the observation at control level $i$, signal level $j$, noise level $k$, and replicate $l$ and $\epsilon_{ijkl} \sim N(0, \sigma_i^2 M_j^2)$. We want to estimate $\eta_i = \log \frac{\beta_i^2}{\sigma_i^2}$. The MLEs of $\beta_i$ and $\sigma_i^2$ are,

$$\hat{\beta}_i = \frac{1}{JKL} \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{y_{ijkl}}{M_j} = \overline{u}_{i...}$$

and

$$\hat{\sigma}_i^2 = \frac{1}{JKL} \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{l=1}^{L} (u_{ijkl} - \overline{u}_{i...})^2$$

where $u_{ijkl} = y_{ijkl}/M_j$. Then the MLE of $\eta_i$ is given by $\hat{\eta}_i = \log \frac{\hat{\beta}_i^2}{\hat{\sigma}_i^2}$. The bias in the estimate can be obtained using Taylor's series expansion as

$$E(\hat{\eta}_i - \eta_i) = -B(\eta_i) + c,$$

where

$$B(\eta_i) = \sum_{n=1}^{\infty} e^{-n\eta_i} \frac{(2n-1)!!}{n(JKL)^n}$$

and

$$c = \log \frac{JKL}{JKL-1} + \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-1)^m \begin{pmatrix} n \\ m \end{pmatrix} \frac{(JKL+2m-3)!!}{n(JKL-3)!!(JKL-1)^m}.$$

The notation $n!!$ means $1.3.5 \cdots n$ when $n$ is odd and $2.4.6 \cdots n$ when $n$ is even. Because $c$ is a constant, we need to consider only the term $B(\eta_i)$. In typical experiments $J \geq 2, K \geq 2$, and $L \geq 1$. Also $\beta_i >> 2\sigma_i$ for the normality assumption to be valid. $B(\eta_i)$ decreases with increase in $JKL$ and $\beta_i/\sigma_i$. Hence the maximum value of $B(\eta_i)$ can be obtained by setting $JKL = 4$ and $\beta_i/\sigma_i = 2$. We obtain that it accounts for only 5% of the $\eta_i$ and therefore the bias can be neglected.

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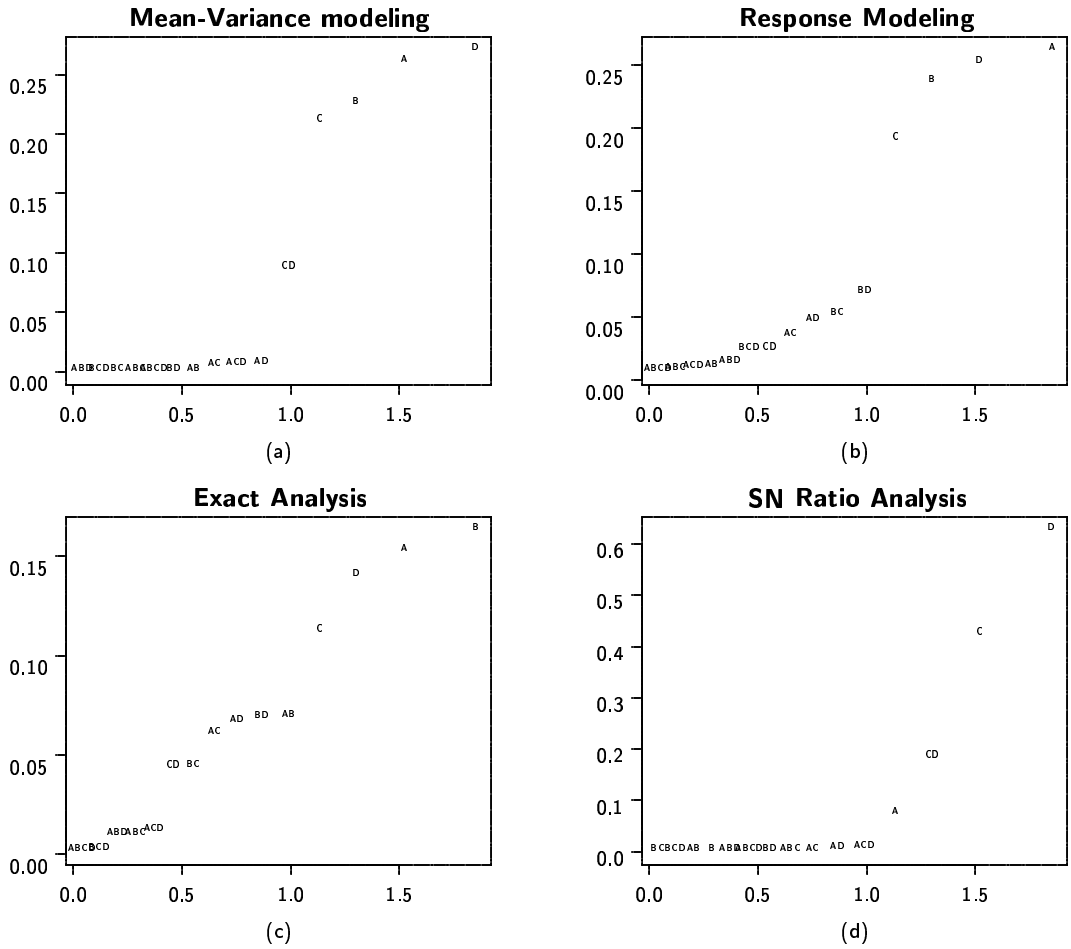


Figure 3: Half-normal plots

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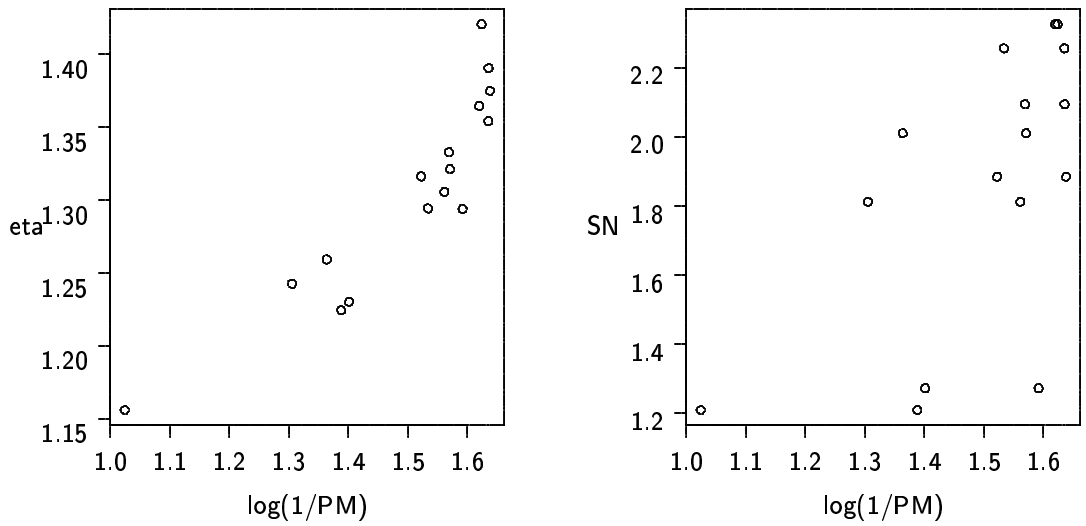


Figure 4: Plots of $\eta$ and $SN$ against $\log(1/PM)$

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