In this lecture of the expanders, we first introduced a method to reduce the error rate of any randomized algorithm A by using the same amount of random bits. The method relies on the existence of a certain "expanding" regular bipartite graph, which is called "magical graph": Let $G = (L, R, E, d), |L| = |R| = n$ be a bipartite graph and each node on the left has exactly d edges going out. It is called magical if the following conditions holds:

- Any $|S| \leq \frac{n}{10d}, S \subseteq L$, $|\Gamma(S)| > \frac{5d}{8}|S|$
- Any $\frac{n}{10d} \leq |S| \leq \frac{n}{2}, S \subseteq L$, $|\Gamma(S)| \geq |S|$

It can be proved that for large d, such graph always exists. This graph makes it possible to reduce the error rate deterministically. The details about this method can be found in the first chapter of the lecture note written by Wigderson et al. (http://www.math.ias.edu/~boaz/ExpanderCourse/)

Then, we moved to the definition of the expander graphs, the edge expansion ratio and the spectral analysis of the d-regular graph. Let $\lambda_1 \geq \lambda_2 ... \geq \lambda_n$ be the sorted eigenvalues of the adjacency matrix of any d-regular graph G. Let $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$ which is the second largest singular value. We say that a d-regular graph G is of the class (n, d, r) if $|V| = n$ and $\lambda = r$.

Definition. Edge expansion ratio $h(G) = \min_{S \subseteq V, |S| \leq \frac{n}{2}} \frac{|E(S,\bar{S})|}{|S|}$, where $E(S, \bar{S}) = \{e = (u, v) \in E | u \in S, v \in \bar{S}\}$.

Theorem. *(Spectral gap)* $\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{d(d - \lambda_2)}$.

From this theorem we know that if the spectral gap is large then the graph is a good expander.

Proposition. *Some combinatorial facts from the analysis of the eigenvalues:*

(1) $\lambda_1 = d$
(2) G is connected $\iff$ $\lambda_2 \neq d$ (Perron-Frobenius Theorem)
(3) G is bipartite implies $\lambda_n = -1$
(4) G is connected and $\lambda_n = -1$ implies that G is bipartite

Finally, we proved the expander mixing lemma:

Lemma. *For any $S, T \subseteq V$, we have $|\ |E(S,T)| - \frac{d|S||T|}{n}\ | \leq \lambda\sqrt{|S||T|}$, where $\frac{d|S||T|}{n}$ is the expected number of edges going from S to T assumed that we are in the purely random case. So if λ is small, the graph G is "like" a general random graph.*

From this lemma, we can say a lot about the underlying graph G, for more details please read the chapter 3 of the lecture note written by Wigderson. One of the implications we did is that the diameter of the graphs of type $(n, d, \frac{d}{2})$ is $O(\log n)$. Actually, it can be proved that the diameter of the graphs of type (n, d, λ) with $\lambda < 1$ is $O(\log n)$. The proof goes like the followings.

Denote A as the normalized adjancency matrix of G, that is the adjacency matrix divided by the degree.

$\lambda = \max_{||x||=1, x \perp u} ||Ax||$, the second largest singular value of A, and in this case, because A is symmetric, $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$.

Theorem. *For any (N, d, α) graph with $\alpha < 1$, there exists an $\epsilon > 0$, s.t. for any $S \subseteq V, |S| \leq \frac{|V|}{2}$, $\Gamma(S) \geq (1+\epsilon)|S|$.*

Proof. $||A\pi - u|| \leq \alpha||\pi - u||$, where π is any probability distribution on the vertices, and u is the uniform distribution $(\frac{1}{n}, ..., \frac{1}{n})$. The fact we use here is $\langle \pi - u, u \rangle = 0$. This essentially proved that the random walk in this kind of graphs is rapid mixing.

By the Pythagoras theorem, $||\pi||^2 = ||\pi - u||^2 + \frac{1}{n} \Rightarrow ||A\pi||^2 \leq \alpha^2||\pi - u||^2 + \frac{1}{n} \leq \alpha^2(||\pi||^2 - \frac{1}{n}) + \frac{1}{n}$.

By the Cauchy's inequality, we have $||\pi||^2|supp(\pi)| \geq 1$, $||A\pi||^2 \geq \frac{1}{|supp(A\pi)|}$.

By the nature of the graph, we have $|supp(\pi^T A)| = |supp(A\pi)| = |\Gamma(supp(\pi)|$.

Now let $p \leq 0.5, \pi$ be the uniform distribution on any set $S \subseteq V$, and suppose $|S| \leq pn$, then we have $\frac{1}{|\Gamma(S)|} - \frac{1}{n} \leq \alpha^2(\frac{1}{|S|} - \frac{1}{n}) \Rightarrow |\Gamma(S)| \geq (\frac{1}{(1-p)\alpha^2 + p})|S|$. Then if $\alpha < 1$, we proved the theorem. □

Proposition. *The diameter of every graph in the graph class (N, d, α) with $\alpha < 1$ is $O(logN)$, because when $l = log\frac{|V|}{2}/log(1+\epsilon)$, from a point s, the number of vertices within the radius of l will exceed $|V|/2$.*

Suppose we start from S, then we get a bigger set $\Gamma(S) \cup S$, s.t. $|\Gamma(S) \cup S| \geq (1+\epsilon)|S|$. So suppose $(1+\epsilon)^l \geq \frac{1}{2}$, we can find the minimum of l satisfying this inequality.