ACO Student Seminar: Open Problems in Graph Theory

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1 Introduction

There are no shortage of graph theory problems. The question is figuring out which ones will be appreciated and extended by others. We will ignore random graphs in this talk in the interest of time and so that other speakers within Georgia Tech may discuss this topic.

2 Ramsey Theory

Recall Ramsey's theorem which loosely states that if r is big enough, and you color $E(K_r)$ using two colors, there is a monochromatic K_n. Let $r(n)$ be the smallest such r. We know that $\sqrt{2} \leq r(n)^{1/n} \leq 4$.

Conjecture 1 *The expression* $\lim(r(n))^{1/n}$ *exists.*

If it exists, what is it? Can we generalize this?

3 Extremal problems

For example, the expression $ex(n, H) = \max\{|E(G)| : |V(G)| = n, H \not\subseteq G\}$. Recall Turan's theorem, which computes $ex(n, K_r)$. Here the optimal solution partitions the vertices in $r-1$ classes to form a complete $r-1$ partite graph, where the partitions are as close to equal in size as possible.

Theorem 2 *(Erdos-Stone-Simonovits)*

$$\lim_{n \to \infty} \frac{ex(n, H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}.$$

This is useful when $\chi(H) \geq 3$. However, what if $\chi(H) = 2$? We know that it is less than n^2, but the theorem does not say anything more.

Conjecture 3

$$ex(n, K_{t,t}) \geq c_t n^{2-1/t}$$

Turan's theorem is not known for hypergraphs. In fact, it the theorem is still open for 3-uniform hypergraphs.

Conjecture 4 *(Turan)*

$$\lim_{n\to\infty} \frac{ex(n, K_4^3)}{\binom{n}{3}} = \frac{5}{9}.$$

4 Graph Minors

Recall that a minor of G is obtained by taking subgraphs and contracting edges.

Theorem 5 *(Mader) Let $n = |V(G)|$. For $t \leq 7$, if $|E(G)| > (t-2)n - \binom{t-1}{2}$, then G has a K_t minor.*

Note that the theorem is tight in all cases where the theorem holds, but the theorem fails for $t = 8$ by considering K_{10} minus a perfect matching. This has 40 edges, but no K_8. In general, at least $ctn\sqrt{\log t}$ edges to guarantee a K_t minor. Andrew Thomason even calculated the constant for this.

Conjecture 6 *(Seymour & RT) If G is $t-2$ connected, $|V(G)| \geq f(t)$ and $|E(G)| > (t-2)n - \binom{t-1}{2}$, then G has a K_t minor.*

Note that the connectivity bound is tight and that the $f(t)$ is required by Thomason's theorem (which is based on random graphs).

5 The k-Disjoint Paths Problem (kDPP)

Consider the following problem: Input: $G, s_1, s_2, \ldots, s_k, t_1, t_2, \ldots t_k \in V(G)$. Question: Do there exist k disjoint paths $P_1, \ldots, P_k$ such that P_i has endpoints s_i and t_i?

Definition 7 *G is k-linked if every kDPP in G is feasible.*

Conjecture 8 *(Thomassen) If G is $(3k-2)$-connected, then G is k-linked.*

This is best possible via the following construction. Take a K_{3k-1}, delete k edges. Call the ends s_i, t_i. This problem is infeasible as there are $k-1$ leftover vertices, but k connections to make. It is known (Wollan) that if G is $10k$-connected, then G is k-linked.

Conjecture 9 *For $k = 3$, if G is 8-connected, then G is 3-linked.*

It would be nice to know the set of all obstructions for graphs that are 3-linked. Note that Robertson and Seymour have an $O(n^3)$ algorithm, but the constant is quite huge. Brute force is actually better than this algorithm for graphs in applications. It would be nice to improve the Robertson and Seymour algorithm, even for special cases.

6 The Four Color Theorem

Recall the four color theorem says that every planar graph is 4-colorable. One problem is to find a non-computer proof.

Conjecture 10 *(Hadwiger, 1940's) If G has no K_{t+1} minor, then it is t-colorable.*

There is some suspicion that this is false. One possible source of a counterexample might be random graphs, but the Erdos-Renyi model doesn't seem to work. Could there be other models that are more appropriate?

Consider the conjecture for various special cases. For $t = 2$, no K_3 minor (no cycles) implies that G is 2-colorable. This is true because no cycles implies that G is bipartite. For $t = 3$, no K_4 minor implies that the $\chi(G) \leq 3$. This can be shown by analyzing graphs known as series-parallel graphs. For $t = 4$, this is equivalent to the four color theorem. A theorem of Robertson, Seymour and Thomas says that the $t = 5$ case also implies the four color theorem.

There are unresolved special cases. The conjecture is open for graphs where $\alpha = 2$, that is there is no independent set of size 3. When $\alpha = 2$ this means $\chi(G)$ is at least $\lceil \frac{n}{2} \rceil$. Hadwiger's conjecture implies that there is a $K_{\lceil \frac{n}{2} \rceil}$ minor. The best known is $K_{\frac{n}{3}}$. In general, we know that no K_t minor implies that the graph is $ct\sqrt{\log t}$-colorable.

The four color theorem is equivalent to the statement that every 2-connected 3-regular planar graph is 3-edge colorable.

Definition 11 *Define a nowhere-zero $\mathbb{Z}_k$-flow as follows. First, direct the edges of G arbitrarily. A flow is a map $F : E(\overrightarrow{G}) \to \mathbb{Z}_k - \{0\}$ such that for all v, the flow into v is equal to the flow out of v in $\mathbb{Z}_k$.*

This is dual to coloring. In other words, G is k-colorable if and only if G has a nowhere-zero $\mathbb{Z}_k$-flow. This is equivalent to saying that G is k-face colorable. To see this equivalence, color the faces with $\mathbb{Z}_k$. Color each edge e as follows: $f(\overrightarrow{e}) = a - b$. We claim this is a nowhere zero $\mathbb{Z}_k$-flow. Notice first that $a \neq b$. The conservation condition holds as the flow around a point is zero.

We may also consider flows in arbitrary graphs. In particular, consider the following three conjectures.

Conjecture 12 *The 3-flow conjecture. For every* 2*-connected graph* G*, if* G *is 4-edge-connected, then it has a nowhere zero* $\mathbb{Z}_3$*-flow.*

Conjecture 13 *The 4-flow conjecture. Every* 2*-connected graph* G *with no Petersen minor has a nowhere zero* $\mathbb{Z}_4$*-flow.*

Conjecture 14 *The 5-flow conjecture. Every* 2*-connected graph has a nowhere zero* $\mathbb{Z}_5$ *flow.*

Comments: Note that the 5-flow conjecture is best possible as the Petersen graph doesn't have a nowhere zero 4-flow. There is a theorem of Seymour that proves the analogous 6-flow theorem. The 4-flow conjecture implies the four color theorem. The 3-flow conjecture is derived from a theorem of Grotsch which states that every triangle-free planar graph is 3-colorable. The dual of Grotsch says that every 2-connected planar graph with no edge cuts of size 3 has a $\mathbb{Z}_3$-flow. So Grotsch's theorem is the same as the 3-flow conjecture for planar graphs.

Conjecture 15 *(Thomassen) There exist* c^n 3*-colorings of triangle-free planar graph* G*, where* c *is a constant and* n *is the number of vertices of* G*.*

An interesting fact is that we can't remove the planar restriction for Grotsch's theorem, but in the dual we can remove the restriction and the theorem still holds. This is a known phenomenon for graphs, and one such example is the Lucchesi-Younger theorem.