The Instability of Instability of Centered Distributions[*,†]

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In memoriam James V. McConnell, 1925–1990

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Abstract

Democratic simple majority voting is perhaps the most widely used method of group decision making in our time. Standard theory, based on "instability" theorems, predicts that a group employing this method will almost always fail to reach a stable conclusion. But empirical observations do not support the gloomy predictions of the instability theorems. We show that the instability theorems are themselves *unstable* in the following sense: if the model of voter behavior is altered however slightly to incorporate any of several plausible characteristics of decision-making, then the instability theorems do not hold and in fact the probability of stability converges to 1 as the population increases, when the population is sampled from a centered distribution. The assumptions considered include: a cost of change; bounded rationality; perceptual thresholds;a discrete proposal space, and others. Evidence from a variety of fields justifies these assumptions in all or most circumstances. One consequence of this work is to render precise and rigorous, the solution proposed by Tullock to the impossibility problem.

All of the stability results given here hold for arbitrary dimension. We generalize the results to establish stability with probability converging to 1 subject to tradeoffs between the assumptions and the degree of non-centeredness of the population. We also extend the results from Euclidean preferences to the more general class of intermediate preferences.

1 Introduction

A principal goal of political economy has been to understand and explain the contemporary processes by which group decisions are made. In present times, democracy based on simple majority rule has emerged as perhaps the predominant mode of group decision making. Standard pure theory predicts the failure of a majority-rule democracy. Indeed, the classic results on social choice are predominantly pessimistic: the counterintuitive voter paradox of Condorcet, the extraordinary impossibility theorem of Arrow, the instability theorems of McKelvey, Rubenstein, Schofield, and others. In particular, the instability theorems predict that majority-rule decision making will not reach a conclusion and can traverse any point in the decision space.

However, instability is not characteristically observed empirically, though in theory it should occur most or all of the time [74]. As Ledyard observes, this "means that the theory as we know it is fundamentally flawed"[37]. This and other gaps between the theory and the observation of democratic decision-making processes have inspired significant alterations in research, such as the literature of "institutionalism," which typically takes as given various sorts of structure or constraints placed on the decision-making process. In this paper we show that a more modest departure from the pure theory suffices to bridge the gap between instability and stability. This is in the same general vein as [37], which views the retreat from the pure theory as premature. Some advantages to our approach are the more generic nature of the results, and the natural linkage with measurable preference data (Section 5). We should also point out that some of the departures here are completely in line with institutionalism (e.g. Model II).

The key idea is to enrich the social choice model by incorporating real characteristics of decision-making behavior. It will turn out that the slightest bit of any of a number of plausible, broadly justifiable model refinements is enough to banish instability. One refinement we consider is a *cost of change*. For example, no change in legislative statute, policy, procedure, or resource allocation can be effected without some cost. Moreover, there is both experimental and physiological evidence that a person can not change his mind without cost. Other refinements that we consider include bounds on the computational ability of the voters, and bounds on the degree of accuracy to which an alternative may be specified. For example, budget amounts may not be in fractions of a cent, or tax rates may not be specified to within more than four decimal places of accuracy.

We explore the societal implications of this individual behavior. Our starting point is the Downsian or spatial model of voting [16, 7, 14, 12] under Euclidean or quadratic concave preferences. This has long been the most widely used model of group choice and has found extensive empirical application as well [52, 51, 48, 47, 50, 49, 30, e.g.]. Current theory, in the form of the classic instability theorems [45, 39, 58, 40, 42, 5], states that instability will occur, leading to chaotic movement and cycling.

Continuing to the main point (section 3), we make an additional assumption regarding the distribution of voter ideal points: the population is drawn according to a centered distribution which is continuous or has bounded support. In section 4 we will show how this assumption may be relaxed to permit non-centered distributions. For now we note that most distributions in the literature are centered, e.g. the multivariate normal, uniform cube, ball, parallelopiped, or ellipsoid. In addition the U.S. Senate data in [32] appears consistent with a centeredness assumption.

We then develop several modifications to the standard Downsian model, each based upon a plausible assumption about decision-making. It turns out that that just the slightest alteration to the basic model, by injection of the least bit of any of these assumptions, nullifies the instability theorems. More precisely, the probability of stability will converge to 1 as the population grows.

The first modification (Model I , section 3.1) assumes that there exists $\epsilon > 0$, such that a voter will prefer a new alternative to the status quo only if the former is at least ϵ better than the latter. This assumption is justified in many situations where there exists some minimal (threshold) nonzero cost $\epsilon > 0$ of change. Often this is a direct economic cost; we will argue that in many other situations the cost of change may be nonzero because of the cost of the greater uncertainty of the new alternative, or for other indirect reasons. Thus Model I applies to a broad variety of situations. Theorem 1 proves that in Model I, however small ϵ, stability ensues for any dimensionality with probability 1 as the population increases. This modification is analogous to the notion of an ϵ-core [75, 60, 34] (or quasi-core [59]) in game theory, and to the concept of an ϵ-equilibrium, and in these terms Theorem 1 guarantees (probabilistically) the existence of an ϵ-core for all positive ϵ.

In related work, Kohler [33] and Bräuninger (using a more accurate computational method) [8] have numerically simulated Model I in two dimensions and obtained results consistent with the general results proved here. Methods to compute the ϵ-core in two dimensions to within a desired order of precision are discussed in [69, 8]. Statistical consistency of the ϵ-core in all dimensions is shown in [69]. The author has also proved that the ϵ-core enjoys a dynamical convergence property: Let r and c be the yolk radius and center, respectively; if $\epsilon > 2r$ a sequence of majority-rule votes under Model I starting at x will always reach an ϵ-core point in at most $\lceil \frac{||x-c||-r}{\epsilon-2r} \rceil$ steps [70].

The concept of *bounded rationality* [61] provides an alternative justification for the $\epsilon > 0$ assumption of Model I (section 3.1.2). If the voters have limited computational power, then they will not be able to recognize arbitrarily small differences in distance (utility). Thus Model I applies, and stability ensues with probability 1 as the population increases. This result mirrors results in game theory, where more cooperative group behavior emerges under the same assumption [44].

When this justification of Model I is invoked, we have the intriguing outcome, that individual imperfections or limitations may contribute to social stability.

Model II (section 3.1) assumes that for some $\epsilon > 0$ no alternative at distance less than ϵ from the incumbent or status quo may be introduced. As with Model I we argue that this assumption is valid in a great many situations. For example, people often are limited by perceptual thresholds, and are unable to distinguish between arbitrarily close alternatives. This assumption is also consistent with the idea of "institutional" restrictions. When this assumption is valid, Theorem 1 again provides a positive outcome (theorem 1): stability with probability 1 as the population grows.

Model II also serves as a precise formulation of Tullock's controversial proposed solution to the instability problem [72, 73]. Forty years ago, Tullock had sought to explain the "irrelevance" of the Arrow impossibility theorem via the 2-dimensional Downsian model. Assuming a uniform rectangular distribution, and a prohibition against "hairsplitting", Tullock argued informally that stability would ensue. Succeeding work by Arrow [3], McKelvey[39, 40], and others [13, e.g.] appeared first to prove, then to disprove, Tullock's ingenious argument. However, Tullock's argument has never been proved nor disproved [71]. We prove it here in Theorem 1, and extend it to arbitrary dimension.

The third and fourth models we consider assume some kind of *a priori* advantage on the part of the incumbent or status quo. Again we argue that this assumption is frequently valid, whether due to the control of proxy votes by the incumbent board at the stockholder's meeting, greater name recognition of political incumbents, or interests vested in the status quo. Theorem 3 (section 3.3) guarantees that in Models III and IV, stability will exist with probability 1 as the population increases. The result confirms work of Sloss[64] and Slutsky[65], who find that if a proposal is *a priori* guaranteed a number of votes, it is easier for it to be a stable point ("easier" here means that the conditions for stability are mathematically weaker, though they are much more complex [31, 6]). Also related is work by Feld and Grofman [20], who study cases in two dimensions with 3 and 5 voters, and observe that incumbency advantage makes stability more likely. Theorem 3 provides a precise and rigorous generalization of their observation.

The last two models we consider are based on *discretization* assumptions. Model V assumes that the distance or utility function calculations are not performed over a continuum, but rather over a discrete set of possible values. This assumption would be valid for instance, if numerical calculations were performed in fixed precision, or if costs were computed only to the nearest cent (or centime). Model VI assumes that the proposal space is a discrete set. These models are similar but not identical to models I and II, respectively. The outcome is identical: Theorem 4 tells us that stability exists with probability 1 as the population increases.

Thus the instability theorems *are themselves unstable* with respect to slight realistic modifications to the basic spatial model. This may help explain why majority rule usually works in practice.

The reinstatement of stability is in the same spirit as in [54], where models of auction participant behavior are enriched to improve consistency between the predicted and actual outcomes of auctions for U.S. Forest Service timber (see also [55, 53, 27]). For another result in the same spirit, consider a famed "negative" result in the theory of voting under incomplete information, due to Austen-Smith and Banks, that informative behavior is not an equilibrium in the Condorcet Jury Theorem framework [4]. Laslier and Weibull have proved that this "instability" result is not stable in the sense that it breaks down if one instills some reasonable perturbation in the description of the voter's decision-making process [36]

While the centeredness of underlying distributions remains an open question, we doubt that real populations or groups behave as though sampled from a perfectly centered distribution (although there is some empirical evidence that the underlying distribution may not be very much off-centered). The ϵ-stability Theorems 1,3, and 4 just described make the strong assumption that the distribution is centered. On the other hand, these theorems make the relatively weak assumption that $\epsilon > 0$. It seems likely that in many of the situations discussed, ϵ will not just be nonzero, but will have non-negligible magnitude. For example, the cost of change or of uncertainty can be substantial in many cases. This suggests that we seek a trade-off between the centeredness of the distribution and the magnitude of ϵ.

The yolk radius and the Simpson-Kramer min-max majority are two measures of non-centeredness. We show (section 4) how a tradeoff between the degree of centeredness and the value of ϵ in Models I–IV leads to stability. When ϵ is greater than the distributional yolk radius, in model I, stability ensues with probability 1 as the population grows (Theorem 6). Similar results hold for the other models. However, we do not wish to conceal the importance of the centeredness assumption, or, as we show in section 6, the related but weaker assumption of intermediate preferences. As one of the referees has pointed out, voting data can be fitted to more than one model of preferences, so what is really at stake is how are the ideal points scattered in a space that makes all individual preferences Euclidean, constant-elasticity-substitution, or intermediate. If they are scattered in a centered or nearly centered way, then the results herein apply.

Ultimately we should like a model of social choice with good explanatory and effective predictive power. Thus we seek stronger tests of our proposed explanation, than that they should merely predict stability. In the case of Model I, there is some very promising corroborative experimental work reported by Salant and Goodstein [57], who formulated Model I independently and prior to the work reported here. These tests and other empirical issues are discussed in section 5. In Section 6 the results are extended to the class of intermediate preferences and some related work on probabilistic models is discussed.

2 Notation and Assumptions

We establish notation and introduce the main assumption of centeredness. The voters form a set $V \subset \Re^m$, where $m \geq 2$ is fixed. Each voter $v \in V$ is independently identically distributed according to a probability distribution μ.

For any hyperplane $h \in \Re^m$, we denote the two closed halfspaces defined by h as h^+ and h^-, respectively, where h^- contains the origin (if h contains the origin the ambiguity won't matter). Let $n = |V|$ denote the number of voters.

Definition: A probability distribution f is *centered around 0* iff for every hyperplane h not passing through 0, the total mass of f in h^+ is less than 1/2.

Definition: A probability distribution $f(x)$ is *centered around* z iff $f(x+z)$ is centered around 0. If f is centered around z we say f is *centered* and z *is the center of* f.

Many common distributions are centered, for example the multinormal (even with singular variance/covariance matrix), or the uniform distribution on a parallelogram or ellipsoid. Most sign-invariant distributions are centered. Note our condition is that the population be drawn from a centered distribution. This is a different and much weaker condition than requiring that the population be centered. See [71] for additional explanation.

Koehler [32] gives some data for the U.S. Senate in 1985. A projection of the 3 dimensional representation is shown in Figure 1. The overall shape is quite interesting: there are two large opposing peaks, and a third, smaller, central peak. On the basis of this data, the group's ideal points are quite polarized and fail statistical fitness tests for both the normal and uniform distribution classes. They also do not appear to be sampled from a concave distribution. But they do appear roughly consistent with sampling from a centered distribution. I do not know of any statistical tests for whether a sample comes from a centered distribution, so I cannot support the preceding statement rigorously. In section 5 some ideas for such statistical tests are discussed, and some of the mathematical groundwork for these is laid here (Corollary 6.2) and in [9, 71]. This area needs some more development.

In addition to the centeredness assumption, μ is taken to satisfy one (or more) of the following technical regularity conditions: (i) μ has bounded support; (ii)μ is continuous; (iii)μ has positive continuous density in a ball containing z. For empirical studies the first of these will ordinarily be satisfied, since we usually take measurements on bounded scales. If we fit this data to a normal or some other continuous distribution, then (ii) will apply. Therefore, the regularity condition is fairly mild, though we dispense with it when possible (e.g. Theorem 7).

3 The Instability of Instability

In this section we introduce several plausible assumptions concerning individual decision-making. Each leads to a modification of the Euclidean model. Theorems 2–4 state that in any of these modified models, there is equilibrium with probability 1 as the population increases, when the population is sampled from a centered distribution.

3.1 Model I: the ϵ-core

The first assumption we consider is that there exists some minimal "threshold" value $\epsilon > 0$ such that agents or voters will select a new alternative in preference to the status quo or incumbent only if the new alternative is at least ϵ closer.

Model I: *Faced with an "incumbent" alternative $a \in \Re^m$ and another alternative $c \in \Re^m$, an agent with ideal point $v \in \Re^m$ will vote for c only if $||a - v|| > ||c - v|| + \epsilon$.*

By "Model I" we actually mean the Euclidean model, modified by the enrichment just stated. It is worth noting that this enrichment is scale-independent, as will be the other models and Theorems 2–4. We rely implicitly on this invariance later in this section, where we argue that Model I will be justified in a great many situations, including some where ϵ may be hard or impossible to measure. All that matters is that ϵ be strictly positive. It should also be pointed out that Model I says "only if". This permits nonuniformity among the agents: each could have a different threshold, as long as no threshold is less than ϵ.

An undominated point in Model I can be viewed as an ϵ-core point in game-theoretic terminology.The ϵ-core clearly is also similar to the notion of an ϵ-equilibrium.

For ϵ sufficiently large obviously the core will be nonempty. When μ is centered, Theorem 1 says the core will become non-empty (w.p.1) as the population increases, for any positive ϵ, however small.

Model II assumes there exists a threshold value $\epsilon > 0$ such that proposals are not distinguished from one another unless they differ by ϵ or more. We define Model II precisely as:

Model II: *There exists $\epsilon > 0$, such that any new proposal c offered to challenge the incumbent proposal a must satisfy $||a - c|| > \epsilon$.*

In Model II the Euclidean model is altered by requiring the "challenger" proposal c to be at distance ϵ or more from the incumbent a. The forbidden points are those meeting Hotelling's principle of adjacency. Then a is an equilibrium point if and only if no point $c, ||c - a|| \geq \epsilon$ is preferred to a by a majority of the voters. The assumption of Model II is mathematically weaker than Assumption I. Nonetheless, Theorem 1 assures eventual equilibrium with probability 1 as the population grows.

Theorem 1. *Let μ be a centered distribution on $\Re^m$ satisfying one of the regularity conditions (i–iii). Let a population of n ideal points with Euclidean preferences be sampled independently from μ. Then in Models I and II, for any $\epsilon > 0$, there is equilibrium with respect to simple majority rule as $n \to \infty$ w.p.1.*

Proof: While the actual proof of Theorem 1 is relegated to the appendix, the intuitive ideas can be seen quite easily in two dimensions.

Consider Figure 2: for an alternative proposal $y \in \Re^2$, the shaded region defined by the hyperbola shown is where y will draw support against an incumbent at the distribution center z. For any particular y, centeredness implies that the probability measure of the associated hyperboloid region is less than 1/2. Then by the law of large numbers, incumbent z will defeat c with probability 1 as the population increases. To prove that *all* alternatives are simultaneously defeated, we employ real analysis to control the probability measures of all the hyperboloid regions with a single bound, and invoke strong results on uniform convergence of empirical measures to be sure this bound is adhered to closely enough in all the regions. $\diamond$

3.2 Motivation for Models I and II

In this section we will show that in the context of Model I, the ϵ-core has a strong foundation in economic and human behavior. We also motivate the assumption of Model II.

3.2.1 The cost of change

A very natural way to justify assumption I is to view ϵ as the cost of change.[1]

There can be very strong economic reasons for resisting change. For example, when the U.S. Congress overhauled the tax law in 1986, the *Wall Street Journal* reported an estimate of 100 billion dollars as the cost to American business for compliance, simply to meet the new legal,

[1]this observation, including the consequent conflict between empirical data and the simple static form of the Von Neumann-Morgenstern utility function, is due to James McConnell.

accounting, and reporting requirements. Thus a change could easily decrease a company's tax liability, yet result in a net loss due to legal, administrative, and data processing costs. In general the cost of altering business plans is enormous. Start-up costs are a major part of business planning for product cycles.

Individuals resist change in many situations. Illustrations of this phenomenon are found in many fields. In marketing science, brand loyalty is a broadly accepted and observed behavior. In political science, party loyalty has been long observed and well documented. Some interesting examples of resistance to change are found in experiments from the study of cognitive dissonance, initiated by Leon Festinger [23]. Festinger found that people with a belief position would reject data in conflict with their beliefs, rather than change their beliefs. Evidence from neurophysiology indicate a physical basis for these observations: there are measurable physiological changes in the brain that accompany learning or changes to memory [38, 29].

The decision-making associated with Model I can be viewed as rational behavior. Consider a utility function $U[p]$ mapping alternatives $p \in \Re^m$ to utility values. In the presence of a cost of change, this simple functional form is inadequate to capture decision-making behavior. A more sophisticated form is needed; e.g. let $U[p_1, p_2]$ denote the utility of being at position p_i in period $i,\ i = 1, 2$. Then we could have $U[a, a] > U[a, b]$ and $U[b, b] > U[b, a]$. Thus whichever of a, b is the present position will be the preferred position for the next period. These preferences would appear reflexive with the less sophisticated functional form[2].

3.2.2 Bounded rationality

The concept of bounded rationality [61] offers another possible justification of Model I. This idea is based on the fact that people have limited or costly computational abilities or resources. Bounded rationality is modelled by assuming computations are limited to those that can be realized on a particular formal model of computation. For example, we may assume as in [44] that each voter's computations are performed on a deterministic finite automaton (DFA). Roughly speaking, a DFA with M states is a computer program consisting of M conditional go-to (branching) statements, but with no memory access permitted (see [63] for a complete exposition). If a voter's computations are limited in this way, it follows that the computations cannot be carried out to arbitrary precision, since they are trivially limited by the number of states, M. If there is a limit on precision, then for some $\epsilon > 0$ (e.g. $2^{-(M+1)}$ if the input is binary), the voter will not be able to distinguish between $||a - v|| - ||c - v|| \leq 0$ and $||a - v|| - ||c - v|| \leq \epsilon$. The exact value of ϵ is not important here because of the scale invariance of the results. Hence computational limitations have the same effect on the Euclidean model as the cost of change, and thus provide an alternate justification of Model I.

Several other formal models of computation lead to the same conclusion as well. For example, suppose the computations are limited to a pushdown automaton (PDA). The PDA is more powerful than the DFA ([63]), so this is a less restrictive assumption. In the appendix we prove that the PDA cannot compute the differences in distances to arbitrary precision, either.

A slightly less formal model would be, any automaton whose arithmetic operations are performed in fixed precision. We summarize with the following proposition:

[2] This form was suggested by Stan Zin and Victor Rios-Rull.

Proposition 2. *If each voter's computations can be performed on a Deterministic Finite Automaton or a Pushdown Automaton or an automaton with fixed precision arithmetic operations, then Assumption I holds.*

We have a philosophical problem here. The Von Neumann-Morgenstern theory specifically does not assume that individuals calculate a utility function. Rather, it asserts that (rational) individuals' decisions can be accurately modelled as though they did. But in the bounded rationality justification of Model I, it seems we implicitly assume voters do calculate their utility functions. Are we making an unrealistically strong assumption about voter behavior?

This problem is resolved through the idea of computation as developed in complexity theory. A person is faced with two alternatives, a and c. If she always decides in a manner consistent with maximizing a utility function $U()$, then she always decides for c when $U(a) < U(c)$, and vice-versa. Thus she is, in effect computing the function $\Delta U(a, c) = U(a) - U(c)$. More precisely, we could say that the voter receives input (a, c) and outputs "No" if and only if $\Delta U(a, c) < 0$. In a formal sense, this input-output description is the definition of computing whether $\Delta U < 0$.

Therefore, we may say that the voter performs computations. We may not say that the voter's computations are performed by some specific method. Nevertheless we can put formal limits on the complexity of these computations by requiring that they could be performed by a specific type of automaton. (This explains why ΔU was used in the proof for PDAs, and why the wording of the Proposition is "... can be performed ...".)

3.2.3 The cost of uncertainty

Assumption I will also be valid in many situations where there is a cost of uncertainty. The uncertainty could be one of two kinds: it could be uncertainty as to what the social decision outcome will be, or it could be an uncertainty as to the consequences of a new alternative.

The first kind of uncertainty can have a large economic cost. For example it is much easier to plan for hard times ahead if one knows whether the problem will be inflation or recession. This is so obvious it seems unnecessary to justify — there is a vast literature on decision-making and planning under uncertainty. If hedging cost nothing, uncertainty wouldn't cause problems. But of course hedging is costly. In fact, planning under uncertainty is the problem which gave rise to linear programming [11]. (This sounds similar to the business cost of change. Here the point is the cost of hedging, such as the cost of a flexible business plan, rather than cost of altering one's business plans.)

The cost of uncertainty on the individual level is illustrated by many psychological experiments in which subjects found the possibility of an undesirable outcome, more stressful than its certainty. One way to interpret this is to assume that the voters or committee members attach a negative utility to not reaching a conclusion. If this utility is at least ϵ in absolute value, then an individual would not propose a new alternative less than ϵ better than the incumbent proposal. This again leads to Model I.

Similarly, positive utility may often be associated with the group's reaching a conclusion. For example, committee members may be permitted to leave a smoke-filled room or go home and eat dinner, provided they reach a conclusion. The purpose of the smoke is to increase the value of ϵ in Model I. The United States Congress always manages to resolve its knotty issues in time for the members to spend Christmas with their families.

The second kind of uncertainty may cause voters to increase their preference for an incumbent, because the incumbent is generally more of a known quantity ("Better the devil you know, than the devil you don't." as the saying goes). This justification would seem more likely to be valid in situations where the social choice is a person, e.g. an elected official. It could also apply to situations where the exact details of the new alternative are unspecified, and are to be worked out later by a negotiating committee or organization. In the United States, the Congressional seniority system may increase preference for incumbents in Senate and House elections. Suppose a voter is indifferent between the incumbent's platform and a challenger's, and favorable overall towards both. The voter may then prefer the incumbent, who by greater seniority would be more effective.

3.2.4 Model II: minimal distance

Model II is partly motivated by situations where people cannot perceive arbitrarily fine distinctions. More precisely, suppose there exists a perceptual threshold value $\epsilon > 0$ such that voters are unable to distinguish between proposals that differ by less than ϵ. Faced with two proposals sufficiently similar to each other, voters will think the two proposals are the same. Therefore any proposed alternative c would have to be at least ϵ distant from the status quo a. (Some evidence of perceptual threshold effects with respect to economic quantities is found in [43].

It is possible to justify Model II from a slightly different point of view. Tullock [72] argues intuitively that the set of proposals that could get a majority over the center would lie in a very small area, if voters were uniformly distributed in a rectangle. He adds, "at this point, however, most voters will feel that new proposals are splitting hairs, and the motion to adjourn will carry." [pages 260–261]. Note that the uniform distribution on a rectangle is centered. Therefore, as promised, one consequence of Theorem 1 is a mathematically rigorous justification of Tullock's "general irrelevance" argument. Tullock's suggestion is not that participants will be unable to distinguish, but that they will not tolerate arbitrarily fine distinctions or "hairsplitting". It seems likely that this assumed limitation of hairsplitting will apply in many situations, simply as a consequence of human behavior.

Model II is also applicable in situations where the institution explicitly constrains new alternatives to be some minimal distance from the incumbent proposal. In this context Model II represents one of the standard institutional methods of restoring stability. Theorem 1 could therefore be employed as theoretical support for this institutional restriction. Theorem 3 and Model IV, to follow, could be invoked to support another standard institutional restriction (supermajority). This aspect of the results is not emphasized because (in the author's opinion) the institutional approach has a more normative flavor than the work here, where the goal is for stability to follow from natural, plausible, widely applicable assumptions (rather than from easily enforced rules).

3.3 Models III & IV: incumbency advantages

The incumbent may enjoy particular advantages in attracting voters, especially in political situations where the incumbent is a person. In the United States, incumbents generally have more visibility and campaign funding than challengers. In many corporate elections the incumbent officers control the proxy votes of inactive small shareholders. In general, there are apt to be interests vested in the status quo, whether the status quo is a person, policy, resource allocation, or law.

Models III and IV are intended to capture the "clout" or other advantage the incumbent may possess. Model III states that the incumbent has an advantage in drawing out supporters.

Model III: *Supporters of the incumbent abstain with probability* $1 - p \geq 0$*; supporters of any challenger abstain with probability* $1 - q > 1 - p$.

Model IV gives the incumbent a bloc of ϵn voters who will vote for her. These might be, for example, uninformed voters who decide on the basis of name recognition, or proxy voters controlled by the incumbent corporate board.

Model IV: *A subset* S, $|S| \geq \epsilon n$ *of voters will vote for the incumbent regardless of their true preferences.*

The key feature of Models III and IV is that the incumbency exerts an influence proportional to the population size. The "ϵ" in these models is the proportionality factor: $(p - q)$ in Model III; the ϵ such that $|S| \geq \epsilon n$ in Model IV. Suppose, for example, 5% of the voting population were uninformed and voted solely on the basis of name recognition, and that a larger fraction of these recognized the incumbent than the challenger. The differential fraction would form a bloc as in Model IV. Theorem 3, following, would imply that an incumbent at the center of the distribution μ would be undefeatable.

Theorem 3. *Let* μ *be a centered distribution. Let a population of* n *ideal points be sampled independently from* μ*. Then in Model III or IV, there is equilibrium as* $n \to \infty$ *w.p.*1.

Proof: Intuitively, nearly 1/2 of the voters will be sure to prefer 0 to the challenger, for large populations. Either modification gives the incumbent a slight additional advantage, which is enough to assure at least half the votes. For a rigorous proof, see the Appendix. (For this proof to apply, the influence need only be more than of order $\sim \sqrt{n}$, but this seems to hold only technical interest). $\diamond$

There is a set of results in the literature on two-party competition that has a similar theme as Theorem 3. This literature studies the stability-inducing effect of non-policy factors on voting behavior in the spatial model. In general, researchers find that incumbency or party identification advantages induce stability. See, for example, [2, 26, 1].

3.4 Discretization: Models V & VI

The last two models we consider involve discretization, that is, limiting the possible values of a variable to a certain discrete set. For example, a computation of present value might be computed in whole dollars.

Model V: Discretized Utilities *For some* $\nu > 0$*, individual decisions are consistent with calculations of utility functions in integer multiples of* ν.

Or, instead of utilities, the set of potential proposals could be discrete. For example, the U.S. president and Congress may have long debates over income tax rates, but all proposals are in whole percents for tax rates and tens of dollars for breakpoints. No one ever suggests a tax rate of 21.462% for taxable income over $34,601.13. Let $\mathcal{Z}$ denote the integers.

Model VI: Discretized Proposals *For some $\nu > 0$, the set of feasible proposals are the lattice points $\{\nu x : x \in \mathcal{Z}^m\}$.*

Theorem 4. *Let μ be a centered distribution satisfying one of the regularity conditions (i–iii). Let a population of n ideal points be sampled independently from μ. Then in Model V or VI, there is stability as $n \to \infty$ w.p.1.*

Proof: Intuitively, Model V should resemble Model I, because any challenger proposal must have utility at least one step higher to be preferred to the incumbent. Model VI should resemble Model II, since both forbid challengers to be arbitrarily close to the incumbent. The rigorous proofs are complicated by the fact that the distributional center or its utility may not be at a discretized value. Details are given in the appendix. $\diamond$

4 Trade-offs between centeredness and ϵ

How realistic is the assumption, that μ is centered? While this remains an open question for empirical study, it seems unlikely to me, *a priori*, that ideal points are distributed as though sampled from a perfectly centered distribution. The purpose of this section is to show how the stability results of the previous sections carry over to the case where μ is not centered. The main result is that there is a trade-off between the degree of centeredness and the magnitude of ϵ in the models.

In Theorems 2–4 of the previous section, the condition on μ is strong (centeredness), while the modifying assumption is weak (e.g. $\epsilon > 0$). Perhaps if μ were *almost* centered, and ϵ were *not too* small, then stability would ensue with high probability.

To make this idea precise, we first need a way to measure how centered or off-centered a distribution is. Two powerful concepts from the literature present themselves: the radius of the yolk, and the min-max majority. The yolk of a configuration of voter ideal points [22, 41] is defined as the smallest ball intersecting all median hyperplanes (each closed halfspace contains at least half the ideal points). The min-max majority [62, 35] is defined as the smallest supermajority that permits some point to be undominated; a point that is so undominated is a min-max winner.

Where V is a configuration of points, $r(V)$ and $c(V)$ will denote the radius and center of the yolk of V, respectively. When V is clear from context we will simply write r and c, respectively. Similarly, the min-max majority is denoted $\alpha^*(V)$ and is always in the interval $[1/2, 1]$. For any supermajority level α, $M_\alpha(V)$ will denote the set of points that are undominated with respect to α-majority rule. Thus M_{α^*} is the set of min-max winners.

We extend the definitions of r, c, α^*, and M to distributions in the natural way (see e.g. [9, 13, 71]). A median hyperplane h of a distribution μ satisfies $\mu(h^+) \geq 1/2$, $\mu(h^-) \geq 1/2$, and the yolk of μ, with radius $r(\mu)$, is the smallest ball intersecting all median hyperplanes. A point x is in $M_\alpha(\mu)$ if and only if, for every hyperplane h containing x, $\mu(h^+) \geq 1 - \alpha$ and $\mu(h^-) \geq 1 - \alpha$.

Both r and α^* are natural measures of centeredness. A distribution μ is centered only if $r(\mu) = 0$, and $\alpha^*(\mu) = 1/2$. (Iff moreover, e.g. when μ is positive continuous.) As r increases from 0, or α^* increases from $1/2$, the distribution becomes more off-centered.

We wish to balance the off-centeredness of a distribution, as measured by r or α^*, against the magnitude of the parameters in the modifications. For finite fixed configurations, the following deterministic theorem gives a trade-off between r and the ϵ of Models I and II. If the cost of change or of uncertainty exceeds the yolk radius, or the perceptual threshold is more than twice the yolk radius, then stability exists.

Theorem 5: *Suppose the yolk radius of a configuration V satisfies $r(V) < \epsilon$, in Model I; or satisfies $r(V) \leq \epsilon/2$, in Model II. Then the yolk center is an undominated point.*

Proof: It suffices to consider the mathematically weaker case of Model II. For any proposal p at distance $2r$ from the yolk center c, the bisecting hyperplane h of segment cp is either tangent to or outside the yolk. Hence at least half the points are in the halfspace containing c, and c is undominated. ■

The probabilistic counterpart to Theorem 5 holds, subject to the same conditions as in Theorem 1. If the cost of change or uncertainty exceeds the distributional yolk radius, then stability exists with probability 1 as the population increases. Similarly, if the distributional yolk radius is smaller than the finest precision of the available computation, or smaller than half the perceptual threshold, then equilibrium exists w.p.1 asymptotically.

Theorem 6: *Let μ be probability measure with distributional yolk radius $r(\mu)$, and let μ satisfy one of the regularity conditions (i–iii). Let a population of n ideal points be sampled independently from μ. Suppose Model I holds with $\epsilon > r(\mu)$, or Model II holds with $\epsilon > 2r$. Then there is stability as $n \to \infty$ w.p.1.*

Proof: See appendix. ◇

We wish to find a result similar to Theorem 6 that applies to Model III. The Simpson-Kramer min-max majority is a natural counterbalance to employ with Model III, since both are multiplicative effects on the voting population. Suppose $\alpha < p/(p+q)$, whence $p(1-\alpha) > q\alpha$. The quantity $p(1-\alpha)$ is a lower bound on the expected fraction of the population which votes for the incumbent; the quantity $q\alpha$ is an upper bound on the fraction voting for the challenger. Thus it is plausible that the incumbent would be undominated. Theorem 7 makes this argument precise.

Theorem 7: *Let μ be a probability measure on $\Re^m$. Suppose $x \in M_\alpha(\mu)$. Let a population of n ideal points be sampled independently from μ. Suppose Model III holds and $\alpha < p/(p+q)$. Then x will be an equilibrium point with probability 1 as $n \to \infty$. (In particular, $\alpha^* < p/(p+q)$ guarantees equilibrium w.p.1 as the population increases.)*

Proof: see Appendix. This proof does not require any of the regularity conditions (i–iii). ◇

5 Empirical issues

We began this study as an attempt to reconcile real-world observations with the theoretical predictions of the spatial model. We have argued for the plausibility and broad justifiability of several slight modifications of the model, and proved that any of these suffices to restore stability. This helps establish the enriched models as legitimate theories to explain observed stability. However, there are several important further steps that must be taken.

First, the results herein are asymptotic. They do not assume an infinite population (see [71] for an explanation of the finite sample method employed here), but they do assume a large population. Often what we would like to know is, for instance, how likely is stability in a population of 9, or 101? This is where the simulation results of Koehler and Bräuninger cited earlier [32, 33, 8] add to the theoretical arguments put forward here.

We now seek stronger empirical validation of our models. Theorems 5 and 6 suggest one kind of test: estimate for example the "cost of change," or precision level, and the yolk radius, and check to see if stability was observed.

There are no statistical tests available to determine whether a sample derives from a centered distribution. These measures suggest the following kind of test: calculate the sample $r(\mu_n)$ or $\alpha^*(\mu_n)$, and compare the value with 0 or 1/2, respectively. The comparison would have to take into account other parameters: n, the sample size, and in the case of r some measure of distance (e.g. the mean distance between sample points) to make the test scale-invariant. Constructing a good test may be difficult or not possible. In any case, a first step would be to establish the consistency of the "estimators" r and α^*. That is, we would like to be sure that

$$\lim_{n\to\infty} r(\mu_n) \to r(\mu) \ a.s. \tag{1}$$

and

$$\lim_{n\to\infty} \alpha^*(\mu_n) \to \alpha^*(\mu) \ a.s. \tag{2}$$

The latter convergence (2) has been established by [9] for continuous bounded functions on domains of compact support, and by [71] in the general case. As a by-product of Theorem 6, Corollaries 6.1 and 6.2 following prove (1), subject to a regularity condition.

Corollary 6.1: *Under the hypotheses of Theorem 6,* $\limsup_n r(\mu_n) \leq r(\mu)$*, a.s.*

Corollary 6.1 is one side of the convergence needed to establish the consistency of the yolk radius statistic. Under additional conditions, the proof also gives the lim inf side of the convergence, which combines with 6.1 to give consistency (1), stated next.

Corollary 6.2: *Under the hypotheses of Theorem 6, and the additional condition that* μ *is continuous and strictly positive in the region of its support,* $\lim_n r(\mu_n) = r(\mu)$*, a.s.*

The additional condition of Corollary 6.2 may seem unnecessarily strong to the reader. However, regularity condition (i) is not sufficient for the convergence (1). A simple counterexample is two points, each with mass 1/2. A simple two-dimensional counterexample is four points at the vertices of a rectangle, each with probability 1/4. In this case we calculate $\liminf_n r(\mu_n) = 1/(2+\sqrt{2}) < 1/2 = \limsup_n r(\mu_n)$. If the rectangle is made into a trapezoid, the ratio between the lim sup and the lim inf can be made as large as desired.

A stronger test would be to check if the societal outcome falls within the "solution set" of the model. If a model passes this stronger test, then it would appear to have good predictive power.

In the case of Model I, experimental work of Salant and Goodstein [57] provides very promising test results. Salant and Goodstein conduct a series of experiments on committee voting under majority rule, and find that the committees often choose alternatives other than the core (undominated point), even when the core coincides with a unique Nash equilibrium in undominated strategies. To reconcile the experimental results with the theory, they modify the model by assuming that agents will only vote for a new alternative if its utility exceeds the incumbent's by at least some nonzero threshold amount. This is of course identical to Model I.[3] With the ϵ-core solution concept Salant and Goodstein reconcile the outcomes of their experiments. They also are able to explain the outcomes of the Fiorina and Plott [24] and Eavey and Miller [17] experiments, which had previously been regarded as structurally different [57, p. 296]. Some other corroborative evidence for Model I is given by Herzberg and Wilson [28], where they find the ϵ-core at least not inconsistent with experimental data. It would be interesting to have similar tests on the other models proposed here.

There is considerable methodological work remaining, as well. Each solution concept brings with it the joint questions of computation and statistical estimation. For the yolk, the algorithm of [68] handles the computational question, and Theorem 6.1 in this paper makes a start at the estimation question. For the Simpson-Kramer min-max set, [69] provides an efficient computational method; as stated the asymptotic consistency is established in [9, 71]. Similar work is called for with respect to the solution concepts of Models I and II. In all these cases, more work remains on estimating variances to incorporate into valid statistical tests and confidence intervals.

6 Extensions and related work

6.1 Extension from Euclidean to Intermediate Preferences

We extend most of the results of the paper from Euclidean preferences to the much more general class of intermediate preferences. This class, introduced by Grandmont [25], is characterized by the "division by hyperplane property": the bisecting hyperplane of segment $\overline{ab}$ separates the voters who prefer a to b from those who prefer b to a. This class includes the C.E.S. (constant elasticity of substitution) utility functions and is a tad more general than the "linear preferences" of [10] (see this article for several nice illustrations of the richness of this class).

All the definitions and models extend without change, except for Models I and V. Model I is special because it is stated in terms of Euclidean distance but Assumption I is about utilities. In the general case we now treat, utilities and distance no longer correspond[4]. We define an extended model as follows, where $U[x]$ denotes the (static) utility of alternative x.

Model I': *For some $\nu > 0$, agents will vote for an alternative c to incumbent proposal a only if they determine that $U[c] > U[a] + \nu$.*

Models I' and V implicitly assume individual preferences are representable by a utility function. To extend the results to these models, we moreover assume continuity of these utilities.

Assumption: ***Continuous Utilities***: *Each voter v has (static) preferences which are representable by a continuous utility function $U_v[a],\ a \in \Re^m$.*

Theorem 8 extends all the results in the paper to intermediate preferences, except for Theorems 5 and 6 in the case of Model I'.

[3] Salant and Goodstein describe their threshold assumption (page 295) as a slight weakening of the rationality hypothesis. As argued in 4.1.1, the threshold assumption can also be viewed as perfectly rational in a great many situations, e.g. in the presence of a cost of change, even if that cost be psychological, or the presence of a benefit to adjournment.

[4] Model I, unmodified, makes mathematical sense. But I don't think it is a meaningful model when preferences are non-Euclidean.

Theorem 8: *Extend Models I',II–VI to intermediate preferences. For Models I' and V, further assume* Continuous Utilities. *Then Theorems 1,2,3,4,and 7 remain true. Theorems 5 and 6 remain true in the case of Model II.*

Proof: The proofs of Theorems 1, 2 (Model II), 3, 4 (Model VI), 5 and 6 (Model II), and 7 all apply to the extended class without change. To prove Theorem 1 for the case of Model I', let $\nu > 0$ be given. By Continuous Utilities at the origin, there exists $\epsilon > 0$ such that

$$||x|| < \epsilon \Rightarrow |U_v[x] - U_v[0]| < \nu \quad \forall v \in V. \tag{3}$$

(We get uniform continuity for free since $|V|$ is finite.)

Under the assumption of the model, no alternative with ϵ of the origin will be selected by any voter. Now we satisfy the condition of Model II, and Theorem 1 for that case applies. The proof of Theorem 4 for the case of Model V is supplemented in precisely the same way[5]. ■

It would be interesting to further extend some of the results, particularly Corollary 1, to even more general classes of preferences, e.g. the continuous class treated in [56]. It is not clear how to formulate such a generalization in a natural yet mathematically proper fashion.

We can weaken the assumption of Model I, to assume that only alternatives at distance greater than ϵ from the status quo are preferred only if they are at least ϵ closer to the ideal point. In this model, the proof of Theorem 1 carries through, under the stronger condition of μ spherically symmetric. Perhaps a different method of proof would yield equilibrium under a weaker condition.

6.2 Probabilistic considerations

There are many reasons to include a probabilistic element in a model of social choice. These include variations in individual behavior, use of data from a sample of the population (e.g. any survey or poll data), uncertainty and inaccuracy of information. The models in this paper are probabilistic, but are nonetheless wedded to the classic deterministic Euclidean spatial model. The probabilistic element enters when V is sampled. Once V is determined, what takes place is deterministic. If we compare the models here with other more deeply probabilistic models, we find some very interesting differences and similarities. For purposes of discussion, we consider a paper on spatial competition by De Palma *et al.*[15], and a related paper on spatial voting by Enelow and Hinich [19]. In [15] Hotelling's principle of minimum differentiation is restored by introducing a "large enough" unobserved probabilistic heterogeneity in consumer tastes; in [19] equilibrium is restored by introducing a "large enough" unobserved probabilistic element to voter utility calculations (see also [18]). In both cases, a competitive equilibrium between two location-seeking entities (firms, candidates) is sought. This contrasts with the goal of a core point in this work.

A more important contrast lies in how probability enters the model. In both [19, 15] the Euclidean rule of selecting the alternative at least distance is altered so that a consumer or voter selects between two alternatives with probabilities weighted by a function of the distances. When the alternatives are equidistant, the probabilities are equal; when the alternatives are almost

[5] Theorems 5 and 6 do not work for Model I' because the yolk radius needs to be compared with the ϵ of (3), which is a function of ν, rather than with ν itself. If we strengthen the *Continuous Utilities* assumption to, say, uniformly Lipschitz continuous with constant K, we easily get extensions of Theorems 5 and 6 to Model I where the yolk radius is multiplied by K.

equidistant, the probabilities are almost equal. This modification eliminates discontinuities in the gain (profit,votes) function [15, pages 771–772], and continuity is essential to the existence of a competitive equilibrium solution to the positioning problem.

In contrast, the probabilistic element invoked here is not sufficient, by itself, to reinstate stability. Indeed, for all of Models I–VI, if $\epsilon = 0$ then there is no stability *a.e.*, as shown in the companion paper [71] (except the 2-D n even case, where the instability is asymptotic). Moreover the instability occurs precisely because of the discontinuity of the associated functional.

Also, the probabilistic assumptions of [15, 19] are ineluctably probabilistic, because no single fixed configuration, in any number of additional dimensions, will deterministically provide the behavior of these models for all pairs of locations chosen by the firms or candidates. Thus the product differentiation in [15] is not due to a fixed though unknown set of additional coordinates. Rather, the customer actions have inherent randomness in that model.

Therefore the models we are comparing are not equivalent, and cannot be made so. Note the results in this paper remain meaningful in the presence of random behavior, because Theorems 2–4,6,7 actually construct a specific point, prior to the sampling, which will be undominated *a.s.*

Despite the inherent differences in the models, there is an important qualitative similarity in the outcomes, that sufficiently large μ, σ^2, and ϵ, respectively, restore equilibrium. In particular, the σ^2 value in [19] is intended to capture some of the same effects as ϵ here. The comparison suggests a question: can the finite sample method be adapted to find a finite population restoration of the principle of minimum differentiation? In the product placement model, the *firms* will probably have a greater cost of change than the consumers, because start-up production costs are typically very high. This issue is also attractive because data on these costs should be readily available; this would make empirical verification more practicable.

7 Appendix: Proofs of stability theorems 1–4, 6–7

The proofs all take the following form:

1. Define a class S of subsets $s_x(\epsilon) \subset \Re^m$, parameterized by $\epsilon > 0$ and $x \in \Re^m, ||x|| = 1$. Members of the class may be halfspaces, for example. To dislodge the incumbent, more than half the population sample must fall into $s_x(\epsilon)$ for some x. Equivalently, the empirical measure μ_n would have to satisfy $\mu_n(s_x(\epsilon)) > 1/2$ for some x.

2. Given arbitrary $\epsilon > 0$, show there exists $\delta > 0$ such that for all x, $\mu(s_x(\epsilon)) \leq 1/2 - \delta$.

3. Verify properties of the class S to invoke convergence theorems of stochastic processes and find that
$$\sup_x |\mu_n(s_x(\epsilon)) - \mu(s_x(\epsilon)| \to 0 a.s.$$
In particular, the greatest deviation of the empirical measure will eventually drop and stay below $\delta/2$, with probability 1. Therefore, given $\epsilon > 0$, there exists $\delta > 0$ such that with probability 1,
$$\forall x,\ \mu_n(s_x(\epsilon)) \leq 1/2 - \delta/2 < 1/2$$
eventually as $n \to \infty$. Combining this with step 1 implies that the incumbent cannot be dislodged as $n \to \infty$ w.p.1.

The class S constructed in step 1 will vary depending on the modification to the model.

We will repeatedly call on Lemmata 1 and 2, following, to accomplish steps 2 and 3, respectively. One preliminary definition will be needed.

Definition. *A probability measure μ on $\Re^m$ is r-centered at 0 iff every hyperplane h at distance $> r$ from 0 satisfies $\mu(h^+) < 1/2$.* When $r = 0$ the distribution μ is centered. More generally r is the radius of a "distributional yolk" centered at 0[22, 67, 71].

Lemma 1. *Let μ be a measure on $\Re^m$, r-centered at 0. In addition let μ satisfy any of the following conditions: μ has bounded support; μ is continuous; μ has continuous positive density in an open ball containing $B(0, r)$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that all hyperplanes h at distance $r + \epsilon$ from 0 satisfy $\mu(h^+) \leq 1/2 - \delta$.*

Proof: If μ is continuous, then $\mu(h_x^+)$ is continuous in x. Since

$$\sup_{||x|| \geq r+\epsilon} \mu(h_x^+) = \sup_{||x||=r+\epsilon} \mu(h_x^+),$$

and the second supremum is taken over a compact set, the supremum is attained at some point $\tilde{x}$, where $||\tilde{x}|| = r + \epsilon$. By assumption of r-centeredness, $\mu(h_{\tilde{x}}^+) < 1/2$. Simply set δ so that $\mu(h_{\tilde{x}}^+) = 1/2 - \delta$. This establishes the lemma for the continuous case.

If μ has bounded support, the idea is to build a polytope containing the ball of radius r, so that each supporting hyperplane of the $r + \epsilon$ ball lies outside one of the supporting hyperplanes of the polytope, within the region of support. See Figure 3 for a 2-dimensional illustration.

For the formal argument, let $S(t)$ denote the sphere of radius t around 0. Let $\mathcal{B}$ be the bounded region of support, which we may take to be a ball around 0 without loss of generality. As before, for any $x \neq 0$ let h_x denote the hyperplane containing x and normal to x. We say *hyperplane h blocks hyperplane k in $\mathcal{B}$* iff $(h^+ \cap \mathcal{B}) \supset (k^+ \cap \mathcal{B})$. In Figure 3, both h_1 and h_2 block k in $\mathcal{B}$.

Consider any $x \in S(r + \epsilon/2)$. Project x out to the point $y = x(r + \epsilon)/(r + \epsilon/2)$ on the sphere $S(r + \epsilon)$. Obviously h_x blocks h_y, regardless of $\mathcal{B}$. Moreover, since $\mathcal{B}$ is bounded (and $\epsilon > 0$), h_x blocks h_z for all z in a neighborhood of y. Thus, for each $x \in S(r + \epsilon/2)$ the hyperplane h_x blocks all h_z for z in a corresponding open set in $S(r + \epsilon)$. Trivially these open sets cover $S(r + \epsilon)$. By compactness of $S(r + \epsilon)$, extract a finite subcover, and recover the corresponding finite set of points $X \subset S(r + \epsilon/2)$. The hyperplanes defined by the points in X form the desired polytope.

Select $\tilde{x}$ to maximize $\mu(h_x^+)$, where the maximum is taken over all $x \in X$. We may do this since $|X|$ is finite. The rest is the same as the last sentence of the proof for the continuous case.

In the last case, the open region between $S(r)$ and $S(r + \epsilon)$ has strictly positive μ-measure δ for some $\delta > 0$. This completes the proof of Lemma 1. ■

We will use Lemma 1 in the case $r = 0$ for step 2 and in the general case for Theorem 6.

Lemma 2. *The class of all closed halfspaces, open halfspaces, (closed) hyperboloids, closed balls, and pairwise unions, pairwise intersections and pairwise set differences and complements of these, enjoy uniform convergence of the empirical measure.*

Proof: This follows from machinery in [46] for producing generalizations of the Glivenko-Cantelli theorem. In particular, Lemma 18 (page 20) implies the class of halfspaces and hyperboloids has polynomial discrimination; Lemma 15 (page 18) lets us augment this class with the complements, and ensuing pairwise unions and intersections. Additional discussion of this methodology is given in [71] Recall that μ_n represents the empirical measure, which places weight $1/n$ at each of the n sample points. That is, let $V = v_1, \cdots, v_n$ denote the sample of n points taken independently from μ. Then $\mu_n(T) = \sum_{i=1}^{n} I_{v_i \in T}/n$.

Theorem 14 (page 18) then tells us that over this large class $\mathcal{S}$ the empirical measure converges to the measure μ:

$$\sup_{S\in\mathcal{S}} |\mu_n(S) - \mu(S)| \to 0 \;\; a.s. \tag{4}$$

■

Now we prove Theorem 1, which states that in Models I and II, a centered distribution will have an undominated point with probability 1, as the population grows.

Theorem 1: *Let n voter ideal points be sampled independently from a centered distribution μ on $\Re^m$, satisfying one of the regularity conditions (i-iii). Suppose simple majority rule is employed, with Euclidean preferences and subject to Model I or II. Then with probability 1, as $n \to \infty$, there is an undominated point with respect to the sample configuration. (Moreover the center of μ can be taken to be that point).*

Proof: Let 0 be the center of μ without loss of generality. We first consider Model I. Let $\epsilon > 0$ be given with respect to Model I. Voter v will select $y \in \Re^m$ over incumbent 0 iff $||v - y|| + \epsilon < ||v||$. (Here $||\;||$ indicates the Euclidean norm.)

Accordingly let

$$s_y(\epsilon) = \{x \in \Re^m : ||x - y|| + \epsilon < ||x||\}$$

If a voter's ideal point is in $s_y(\epsilon)$, the voter will prefer y to the incumbent 0. Thus 0 will be undominated if

$$\mu_n\left(s_y(\epsilon)\right) < 1/2 \quad \forall y \in \Re^m. \tag{5}$$

That is, no alternate proposal y can muster 1/2 of the sample population's support against 0. For any $y \neq 0$, let $h_y(\epsilon)$ denote the hyperplane normal to y, passing through $\epsilon y/||y||$ (*i.e.*, at distance ϵ from 0). See Figure 2. The hyperboloid $s_y(\epsilon)$ is necessarily contained in the halfspace $h_y^+(\epsilon)$. Hence,

$$\sup_{y\in\Re^m} \mu_n(s_y(\epsilon)) \leq \sup_{y\in\Re^m} \mu(h_y^+(\epsilon)) \tag{6}$$

Apply Lemma 1 with $r = 0$. So there exists $\delta > 0$ such that

$$\sup_{y\neq 0} \mu(h_y^+(\epsilon)) \leq 1/2 - \delta \tag{7}$$

The halfspaces $h_y^+(\epsilon)$ are contained in the class $\mathcal{S}$ of Lemma 2 By Lemma 2, as $n \to \infty$, we have[6]

$$\sup_{y\neq 0} |\mu_n(h_y^+(\epsilon)) - \mu(h_y^+(\epsilon))| \to 0 \;\; a.s. \tag{8}$$

In particular, consider the value $\delta/2 > 0$. The convergence implies that as $n \to \infty$,

$$\sup_{y\neq 0} |\mu_n(h_y^+(\epsilon)) - \mu(h_y^+(\epsilon))| < \delta/2 \;\; a.s. \tag{9}$$

[6]Actually, Lemma 2 shows uniform convergence for the class $s_y(\epsilon)$ as well.

Substituting inequality (7) into (9) implies as $n \to \infty$,

$$\sup_{y \neq 0} |\mu_n(h_y^+(\epsilon))| < 1/2 - \delta/2 < 1/2, \ w.p.1,$$

which by inequalities (5) and (6) proves the theorem for Model I.

We now turn to Model II. For notational convenience, let $2\epsilon > 0$ denote the threshold value of Model II. Hence $||y|| \geq 2\epsilon$. The voter v will prefer y iff v is in the halfspace $h_y^+(||y||/2)$. This halfspace is contained in the halfspace $h_y^+(\epsilon)$. Now we are in exactly the same situation as at inequality (6) in the proof of Model I, which completes the proof. ■

The uniform convergence means we may take Ω as the set of all infinite sequences of points in $\Re^m$, with measure derived from μ.

Proof of the claim for Proposition 2: If there existed a PDA that could compute these differences, even in two dimensions, then for arbitrary (a_1, a_2), (v_1, v_2), (c_1, c_2) it could determine if $(a_1 - v_1)^2 + (a_2 - v_2)^2 - (c_1 - v_1)^2 + (c_2 - v_2)^2 \leq 0$. Then under closure properties of the context-free languages (and their equivalence to those accepted by PDAs), there would exist a PDA that could recognize strings of form $\{(w, x, y, z\} : w^2 + x^2 = y^2 + z^2)$. But this contradicts the pumping lemma for PDAs [63]. This establishes the claim.

Theorem 3: *Let n voter ideal points be sampled independently from a centered distribution μ on $\Re^m$. Suppose simple majority rule is employed, with Euclidean preferences modified as in Model III or IV. Then with probability 1, as $n \to \infty$, there is an undominated point with respect to the sample configuration.*

Proof: Consider the class $\mathcal{C}$ of halfspaces $h_0^+(x)$ generated by hyperplanes $h(x)$ through 0, normal to $x \in \Re^m$. Since $\mathcal{C} \subset \mathcal{S}$, Lemma 2 implies that

$$\sup_x |\mu_n(h^+(x))| \leq 1/2$$

For any $\delta > 0$, the incumbent will be preferred by at least $(1/2 - \delta)n$ members of the population, in the limit w.p.1. Let $q > p$ from Model III be given. Select positive $\delta < (q - p)/[4 - 2(p + q)]$. According to Model III, the incumbent will receive at least $\sim (1 - p)(1/2 - \delta)n$ votes; the challenger will receive no more than $\sim (1 - q)(1/2 + \delta)n$ votes. (Here we are implicitly relying on the exponentially small tails of the binomial distribution to substitute the expected number of non-abstaining votes for the random number who will actually vote. — see the proof of Theorem 7 for details.) Algebra shows that $(1 - p)(1/2 - \delta) > (1 - q)(1/2 + \delta)$. Therefore the incumbent receives more votes than the challenger. This establishes equilibrium under Model III.

Similarly, under Model IV, with probability 1 nearly 1/2 the voters prefer the incumbent, and the extra $\Omega(n)$ support is enough to maintain 0 in equilibrium. This completes the proof of Theorem 3. ■

An additional nondegeneracy assumption is required for the proof of Theorem 4. The assumption amounts to the requirement that the location or utility of the center of the distribution does not fall at an exact midpoint between two allowable discrete values. This condition holds with probability 1 if the center of the distribution is independent of the base value of the discretization. Thus it is an exceedingly mild assumption which however we make explicit for the sake of mathematical rigour.

Nondegenerate Normalization Condition. *Let c be the center of the distribution μ. For Model V: the utility functions U_v of the voters v are normalized at c such that $U_v(c)/\nu - \lfloor U_v(c)/\nu \rfloor = \eta \neq 1/2$. For Model VI: the coordinates of the center satisfy $c_i/\nu - \lfloor c_i/\nu \rfloor \neq 1/2$, $i = 1, \cdots, m$.*

Theorem 4: *Let n voter ideal points be sampled independently from a centered distribution μ on $\Re^m$, satisfying one of the regularity conditions (i-iii). Suppose simple majority rule is employed, with Euclidean preferences and subject to Model V (resp. VI) and the corresponding nondegenerate normalization condition. Then with probability 1, as $n \to \infty$, there is an undominated point with respect to the sample configuration.*

Proof (Model V): To start we make the discretized utility function precise. Let the utility function and its discretization be denoted U and DU, respectively. Let DU be found by rounding. Thus if $U[a] = w\nu$ (w not necessarily integer), then $DU[a] = \lfloor w + 1/2 \rfloor \nu$.

Let the center of the distribution be c. Now let $\epsilon = DU[c] + \nu/2 - U[c]$. The nondegenerate normalization condition guarantees $\epsilon > 0$. This transforms the situation into an equivalence with Model I, and the rest follows from Theorem 1.

Proof (Model VI): This case is not immediately equivalent to Model II because we cannot know that the distribution center c is a member of the lattice of admissible proposals. In other words, once we assume 0 is an admissible proposal, we cannot assume 0 is the distribution center without loss of generality. It will be more convenient in the proof to do the converse: we take $c = 0$ without loss of generality, and shift the lattice instead. Formally, let the lattice of admissible proposals be denoted $\mathcal{L} \equiv \{\beta + \nu x : \ x \in \mathcal{Z}^n\}$. Here β is some fixed point in $\Re^m$.

Each coordinate value of β can be taken to be in the interval $[0, \nu)$, since adding integer multiples of ν does not affect the definition of $\mathcal{L}$. The nondegenerate normalization condition ensures in addition that $\beta_i \neq 1/2 \ \forall i$. Define $\bar{c}$, the *discretized center* of μ, as

$$\begin{array}{rcl} \bar{c}_i = & \beta_i & \text{if } \beta_i < \nu/2 \\ & \beta_i - \nu & \text{if } \beta_i > \nu/2 \end{array}$$

Our goal now is to show that the discretized center $\bar{c}$ is undominated. For any other lattice point $y \in \mathcal{L}$, $y \neq \bar{c}$, let $h_y(\bar{c})$ denote the hyperplane bisecting the segment $\overline{y\bar{c}}$ (normal to $y - \bar{c}$, passing through $(y + \bar{c})/2$). Of the two halfspaces defined by $h_y(\bar{c})$, let $h_y^+(\bar{c})$ denote the one containing y. By the triangle inequality, $c = 0$ is closer to $\bar{c}$ than to y. So $0 \notin h_y^+(\bar{c})$. Let $\epsilon > 0$ equal the minimum distance from 0 to $h_y(\bar{c})$, over all $y \in \mathcal{L}$, $y \neq \bar{c}$. This minimum exists because it is attained at some $y : \ |y_i| \leq 2\nu$.

If $\bar{c}$ is the incumbent proposal, each feasible alternative proposal receives support from voters in a halfspace at least ϵ from 0, the true center of the distribution μ. This situation is identical to that of Model II, and the rest follows from Theorem 1. This completes the proof of Theorem 4. $\blacksquare$

Proof of Theorem 6: Take the yolk center $c = 0$ without loss of generality. Let $\Delta = (\epsilon - r)/2$ (respectively $(\epsilon/2 - r)/2$ for the case of Model II.) Given $\Delta > 0$, apply Lemma 1 to find $\delta > 0$

such that all hyperplanes at distance $r + \Delta$ from 0 satisfy $\mu(h^+) < 1/2 - \delta$. Note $r + \Delta < \epsilon$ (respectively $r + \Delta < \epsilon/2$).

By Lemma 2, w.p.1 for all these hyperplanes $\mu_n(h^+) < 1/2 - \delta/2$ eventually as n increases. Then all median hyperplanes of the empirical measure (i.e. the finite sample configuration) are at distance less than ϵ (resp. $\epsilon/2$) from 0. Apply Theorem 5 to complete the proof. ■

As a consequence of the proof, a ball of radius ϵ centered at 0 would intersect all median hyperplanes. This implies that eventually, the sample yolk radius will satisfy $r(\mu_n) \leq \epsilon$. Since this is true for all $\epsilon > r(\mu)$, we conclude that $\limsup_{n\to\infty} r(\mu_n) \leq r(\mu)$ *a.s.*

Now we make the additional assumption of Corollary 6.2, that μ is positive and strictly continuous in its region of support. Then for any $\delta > 0$, there exists $\Delta > 1/2$ such that all the hyperplanes h at distance $r(\mu) + \delta$ from 0 have mass at least Δ in the closed halfspace they define, i.e. $\mu(h^-) \geq \Delta > 1/2$. So by the same argument as above, we conclude that $\liminf_{n\to\infty} r(\mu_n) \geq r(\mu)$ *a.s.* This inequality combined with Corollary 6.1 proves Corollary 6.2. ■

Theorem 7: *Let μ be a probability measure on $\Re^m$. Suppose $x \in M_\alpha(\mu)$. Let a population of n ideal points be sampled independently from μ. Suppose Model III holds and $\alpha < p/(p+q)$. Then x will be an equilibrium point with probability 1 as $n \to \infty$.*

Proof of Theorem 7: To start we simply count supporters, ignoring the issue of abstention. By the convergence of the min-max majority [9, Theorem 3](continuous case), [71, Theorem 6](general case), (this also follows easily from Lemma 2), $x \in M_\alpha(\mu)$ implies that w.p.1, the fraction of population x will muster against all alternatives converges to $1-\alpha$ or more. This does not mean that x will have the support of precisely $n(1-\alpha)$ or more voters (i.e. that x will be an α-majority winner, see [71]). But it does mean that for any $\delta > 0$, $x \in M_{\alpha+\delta}(\mu_n)$ eventually as $n \to \infty$ w.p.1. That is, eventually x will be a $(\alpha+\delta)$-majority winner, in the sample configuration, w.p.1.

In the preceding paragraph, we were counting supporters rather than votes. Now we take abstentions into account. Fix $\delta > 0$ with a value to be determined later. Assume all statements in this paragraph are modified by the phrase "eventually as $n \to \infty$, with probability 1." The incumbent x gets at least $n(1-\alpha-\delta)$ supporters. Therefore x receives a number of votes (stochastically) greater than or equal to a random variable X distributed according to a binomial distribution $B[n(1-\alpha-\delta), p]$ (success probability p, with $n(1-\alpha-\delta)$ trials.) Out of the n members of the population c gets at most $n(\alpha+\delta)$ supporters, and receives votes stochastically less than random variable C distributed as $B[n(\alpha+\delta), q]$.

Our hypothesis is $(1-\alpha)p > q\alpha$. Hence there exists $\delta > 0$ such that $(1-\alpha-\delta)p > q(\alpha+\delta)$. This determines $\delta > 0$ as promised. We now have two binomially distributed variables, X and C, with expected difference proportional to n, namely:

$$[(1-\alpha-\delta)p - (\alpha+\delta)q]\, n.$$

By the Chernoff bounds [66] or other bounds on tail probabilities of the binomial distribution [21], the value of X will exceed the value of C almost surely as $n \to \infty$. This completes the proof of Theorem 7. ■

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