Formulas for Stochastic Processes

Discrete-Time Markov Chains (DTMC)

State space: S; usually $S = \{1, \ldots, N\}$.

One-step transition probabilities: $p_{ij} = \Pr(X_{n+1} = j | X_n = i)$, $P = [p_{ij}]$.

Initial distribution: $a_i = \Pr(X_0 = i)$.

***n*-step transition probabilities:** $p_{ij}^{(n)} = \Pr(X_n = j | X_0 = i)$, $P^{(n)} = P^n$.

Marginal distribution of X_n: $a_i^{(n)} = \Pr(X_n = i)$, $a^{(n)} = a * P^n$.

Occupancy times: $m_{ij}(n) = E(\#$ of visits to state j during $\{0, 1, \ldots, n\} | X_0 = i)$,
$M(n) = \sum_{r=0}^{n} P^r$.

Recurrence times: $T_{ii} = \min\{n \geq 1 : X_n = i | X_0 = i\}$.

$$\text{State } i \text{ is called} \begin{cases} \text{transient} & \text{if } \Pr(T_{ii} < \infty) = \sum_{n=1}^{\infty} \Pr(T_{ii} = n) < 1 \\ \text{recurrent} & \text{if } \Pr(T_{ii} < \infty) = 1 \Leftrightarrow \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty \\ \text{positive recurrent} & \text{if } \Pr(T_{ii} < \infty) = 1 \text{ and } E(T_{ii}) < \infty \\ \text{null recurrent} & \text{if } \Pr(T_{ii} < \infty) = 1 \text{ and } E(T_{ii}) = \infty. \end{cases}$$

Periodicity: State i has period $d(i)$ if $p_{ii}^{(n)} = 0$ when n is not a multiple of $d(i)$ and $d(i)$ is the largest number with this property. If $d(i) = 1$, i is called aperiodic.

Periodicity, transience, positive recurrence, and null recurrence are class properties.

Criterior for positive recurrence: An irreducible Markov chain is positive recurrent if the balance equations

$$\pi_j = \sum_{i \in S} \pi_i p_{ij} \quad (j \in S); \qquad \sum_{j \in S} \pi_j = 1$$

have a unique positive solution. In that case, $\{\pi_j, j \in S\}$ is the occupancy distribution. If the Markov chain is also aperiodic, then $\{\pi_j\}$ is the limiting distribution, that is, $\pi_j = \lim_{n \to \infty} a_j^{(n)}$ (regardless of the initial state).

Stationary distribution ($\boldsymbol{\pi^*}$): If $a = \pi^*$, then $a^{(n)} = \pi^*$ $(n \geq 1)$. If it exists, it satisfies the balance equations.

Expected total cost over a finite horizon:
$c(i) =$ expected cost incurred at every visit to state j,
$g(i, n) = E(\text{total cost incurred during } \{0, 1, \ldots, n\} | X_0 = i)$,
$c = (c(1), \ldots, c(N))^T$, $g(n) = (g(1, n), \ldots, g(N, n))^T$ (column vectors).
$g(n) = M(n) * c$.

Long-run expected cost per unit time: If the occupancy distribution exists, then
$g = \sum_{j=1}^{N} c(j) \hat{\pi}_j$ (regardless of the initial state).

First passage times: $\mu_{ij} =$ mean time until first (re)visit to state j given $X_0 = i$.
$\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} \mu_{kj}$, for all (i, j).

The Exponential Distribution

Density: $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

Comparing independent exponentials: $\Pr\{\text{expo}(\lambda) < \text{expo}(\mu)\} = \lambda / (\lambda + \mu)$.

Minimum of independent exponentials: If $X_i \sim \text{expo}(\lambda_i)$ are independent, then $\min\{X_1, X_2, \ldots, X_n\} \sim \text{expo}(\lambda_1 + \lambda_2 + \ldots + \lambda_n)$.

Strong memoryless property: If $X_i \sim \text{expo}(\lambda_i)$ are independent, then

$$\Pr(X_k - X_1 > x_k; k = 2, 3, \ldots, n | X_k > X_1; k = 2, 3, \ldots, n) = \Pr(X_2 > x_2) P(X_3 > x_3) \cdots \Pr(X_n > x_n).$$

Sum of i.i.d. exponentials: If X_i are i.i.d. expo(λ), then $Z = \sum_{i=1}^{n} X_i \sim \text{Erlang}(n, \lambda)$. The density and c.d.f. of Z are

$$f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \text{and} \quad F(t) = 1 - \sum_{i=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!}, \qquad t \geq 0.$$

Geometric sum of i.i.d. exponentials: If X_i are i.i.d. expo(λ) and N is independent of the X_i's with $\Pr(N = k) = p(1-p)^{k-1}$, $k \geq 1$, then $\sum_{k=1}^{N} X_k \sim \text{expo}(\lambda p)$.

Poisson Processes (PP)

Definition: A counting process $\{N(t), t \geq 0\}$ is a PP(λ) iff the times between successive events are i.i.d. expo(λ).

Increments: The increments $N(s+t) - N(s)$ of a PP(λ) are independent. For fixed t, $N(s+t) - N(s) \sim$ Poisson(λt).

Event times: Let S_n be the nth event time in a PP(λ). Given $N(t) = n$,

$$(S_1, S_2, \ldots, S_n) \sim (U_{(1)}, U_{(2)}, \ldots, U_{(n)}),$$

where $U_{(1)} < U_{(2)} < \cdots < U_{(n)}$ are the order statistics of n i.i.d. uniform random variables on the interval $(0, t)$. In particular, $E(S_k | N(t) = n) = kt/(n+1)$.

Superposition: If $\{N_i(t)\}$ are independent PP(λ_i) then $\{N(t) = N_1(t) + N_2(t) + \cdots + N_r(t)\}$ is a PP$(\lambda_1 + \lambda_2 + \cdots + \lambda_r)$.

Splitting: Suppose $\{N(t)\}$ is a PP(λ) and each event is of type $1, 2, \ldots, r$ with respective probabilities $p_1, p_2, \ldots, p_r$ independently of other events. Let $N_i(t)$ be the number of type i events in $(0, t]$. Then $\{N_i(t)\}$ are independent PP(λ_i).

Nonhomogeneous Poisson process: A NPP $\{N(t)\}$ with rate function $\lambda(t)$ has independent increments with

$$N(s+t) - N(s) \sim \text{Poisson}\left(\int_s^{s+t} \lambda(u)\, du \right).$$

Compound Poisson process: $\{Z_n, n \geq 1\}$ is a sequence of i.i.d. random variables independent of a PP(λ), $\{N(t)\}$. The CPP $\{Z(t) = \sum_{n=1}^{N(t)} Z_n\}$ satisfies

$$E(Z(t)) = \lambda t E(Z_1), \quad \text{Var}(Z(t)) = \lambda t E(Z_1^2) \quad \text{for } t \geq 0.$$

Continuous-Time Markov Chains

Transition probabilities: $p_{ij}(t) = \Pr(X(t) = j | X(0) = i)$, $P(t) = [p_{ij}(t)]$.

Transition rates: $q_{ij} = \lim_{t \to 0} p_{ij}(t)/t$, $i \neq j$.

Sojourn times: Let $S_n =$ time of nth transition ($S_0 = 0$). If $X(S_{n-1}+) = i$, then the sojourn time $Y_n = S_n - S_{n-1}$ in state i is expo(r_i), where $r_i = -\lim_{t \to 0}(p_{ii}(t) - 1)/t$.

Rate matrix: $Q = [q_{ij}]$, where $q_{ii} = -r_i$.

Embedded Markov chain (EMC): The EMC $\{X_n = X(S_n+), n \geq 0\}$ has transition probabilities $p_{ij} = q_{ij}/r_i$, $i \neq j$. $P = [p_{ij}]$.

Transience and recurrence: A class is transient (recurrent) if and only if it is transient (recurrent) for the EMC.

Limiting distribution: The limiting probabilities $p_i = \lim_{t\to\infty} p_{ki}(t)$ can be computed by solving the balance equations

$$p_i \sum_{j\in S} q_{ij} = \sum_{j\in S} p_j q_{ji}, \quad i \in S; \qquad \sum_{i\in S} p_i = 1.$$

Birth-death process: The only nonzero transition rates are the birth rates $q_{i,i+1} = \lambda_i$, $i \geq 0$, and the death rates $q_{i,i-1} = \mu_i$, $i \geq 1$. Let $\rho_o = 1$ and

$$\rho_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}, \quad i \geq 1.$$

The limiting distribution exists if and only if $\sum_{j=0}^{\infty} \rho_j < \infty$ and is given by

$$p_0 = \frac{1}{\sum_{j=0}^{\infty} \rho_j} \qquad \text{and} \qquad p_i = \frac{\rho_i}{\sum_{j=0}^{\infty} \rho_j}, \quad i \geq 1.$$

Long-run expected average costs: When the Markov process $\{X(t)\}$ is in state i, it incurs cost at rate $c(i)$ per unit time. If $\sum_{i\in S} |c(i)| p_i < \infty$, then

$$\lim_{t\to\infty} E\left(\frac{1}{T}\int_0^T c(X(t))\,dt\right) = \sum_{i\in S} c(i) p_i.$$

Renewal Processes

Renewal times $0 = S_0 < S_1 < S_2 < \cdots$.
Cycle times $T_n = S_n - S_{n-1}$, $n \geq 1$, with mean τ.
$C(t)$ = total net cost incurred during $(0, t]$.
$C_n = C(S_n) - C(S_{n-1})$ = cost incurred during cycle n.

Long-run renewal rate:

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\tau} \quad \text{with probability 1.}$$

Long-run cost rate: If the pairs (T_n, C_n) are i.i.d.,

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{E(C_1)}{E(T_1)} = \frac{E(C_1)}{\tau} \quad \text{with probability 1.}$$

Queueing Theory

Notation:

$$\begin{aligned} \lambda &= \text{arrival rate.} \\ \mu &= \text{service rate.} \\ \tau &= \text{mean service time} = 1/\mu. \\ \sigma^2 &= \text{variance of service times.} \end{aligned}$$

Pollaczec-Khintchine formula for M/G/1 queue:

$$W_q = \frac{\lambda(\sigma^2 + \tau^2)}{2(1-\rho)}; \quad \rho = \frac{\lambda}{\mu}.$$

M/M/s/K queue:

$$\rho = \frac{\lambda}{s\mu}$$

$$p_i(K) = p_0(K)\rho_i, \quad 0 \le i \le K,$$

where

$$\rho_i = \begin{cases} \frac{(\lambda/\mu)^i}{i!} & \text{if } 0 \le i \le s-1 \\ \frac{s^s}{s!}\rho^i & \text{if } s \le i \le K \end{cases}$$

and

$$p_0(K) = \left[\sum_{i=0}^{s-1} \frac{(\lambda/\mu)^i}{i!} + \frac{(\lambda/\mu)^s}{s!} \frac{1-\rho^{K-s+1}}{1-\rho} \right]^{-1}.$$

M/M/s queue:

$$\rho = \frac{\lambda}{s\mu} < 1$$

$$p_i = p_0\rho_i, \quad i \ge 0,$$

where

$$\rho_i = \begin{cases} \frac{(\lambda/\mu)^i}{i!} & \text{if } 0 \le i \le s-1 \\ \frac{s^s}{s!}\rho^i & \text{if } i \ge s \end{cases}$$

and

$$p_0 = \left[\sum_{i=0}^{s-1} \frac{(\lambda/\mu)^i}{i!} + \frac{s^s}{s!} \frac{\rho^s}{1-\rho} \right]^{-1}.$$

$$L = \frac{\lambda}{\mu} + p_s \frac{\rho}{(1-\rho)^2}.$$

Open Jackson networks:

$$\begin{aligned}
N &= \text{number of stations.} \\
\lambda_i &= \text{external arrival rate at station } i. \\
\lambda &= (\lambda_1, \dots, \lambda_N). \\
p_{ij} &= \text{routing probability from station } i \text{ to station } j. \\
P &= [p_{ij}]. \\
a_i &= \text{total arrival rate at station } i \\
&= \lambda_i + \sum_{j=1}^{N} a_j p_{ji} \quad (1 \le i \le N). \\
a &= (a_1, \dots, a_N). \\
s_i &= \text{number of servers at station } i. \\
\mu_i &= \text{service rate at station } i.
\end{aligned}$$

If the traffic equations $a = \lambda + aP$ have a solution and $a_i < s_i\mu_i$ $(1 \le i \le N)$, the stations behave independently of each other in steady state, each as an M/M/s_i system.