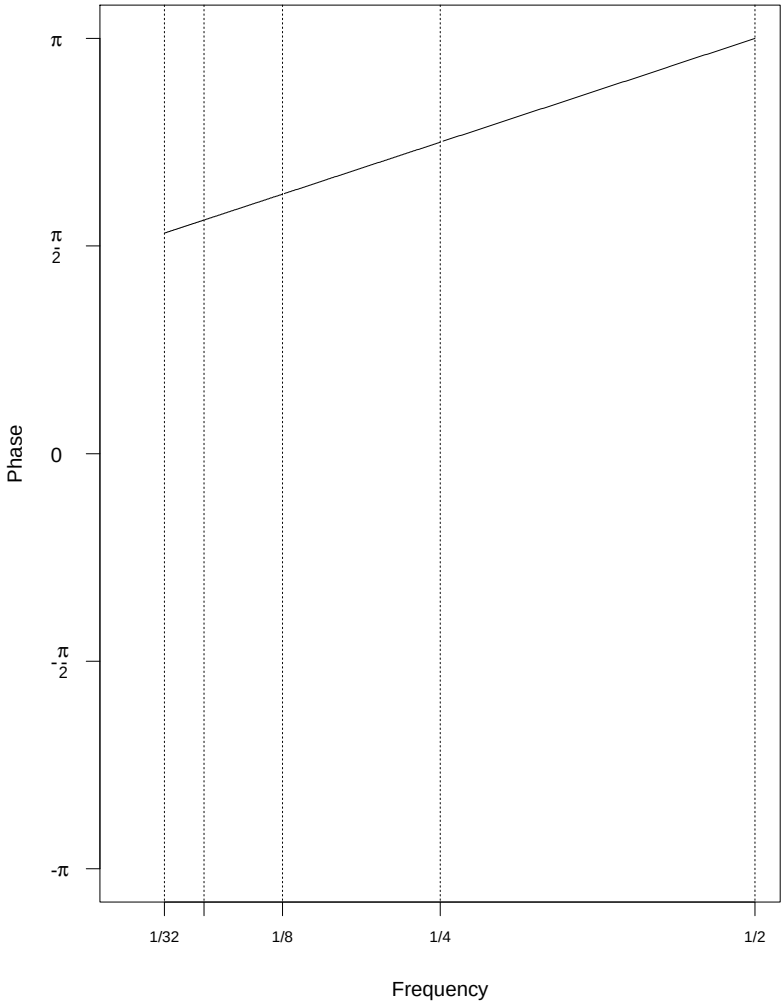

Figure 3.

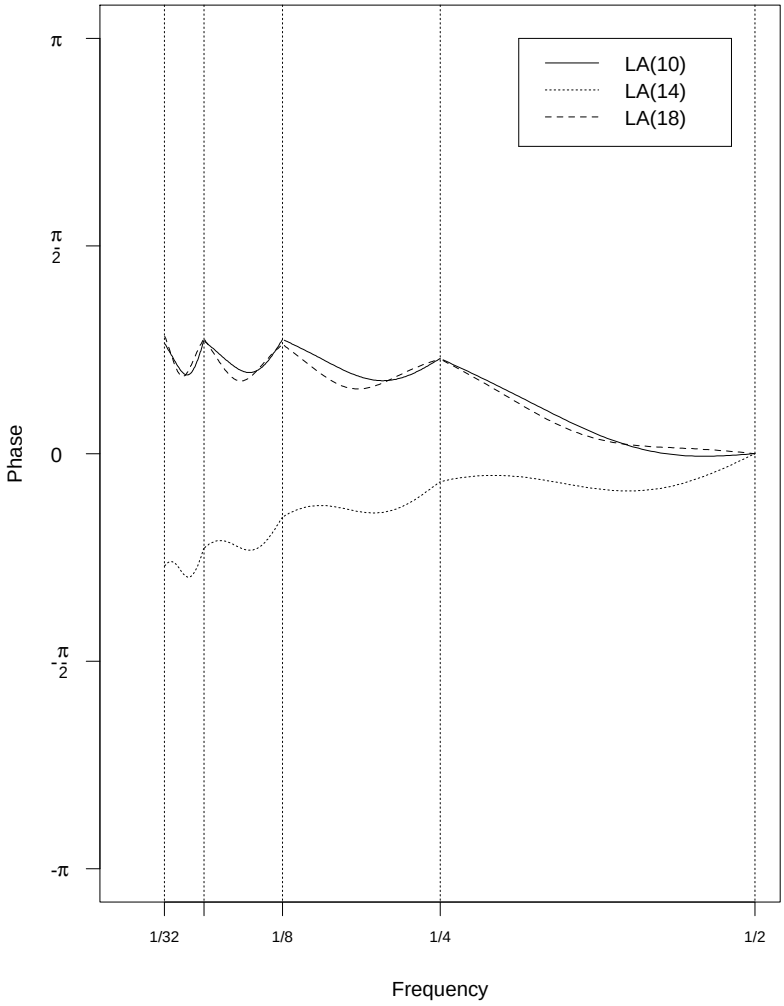

Figure 2.

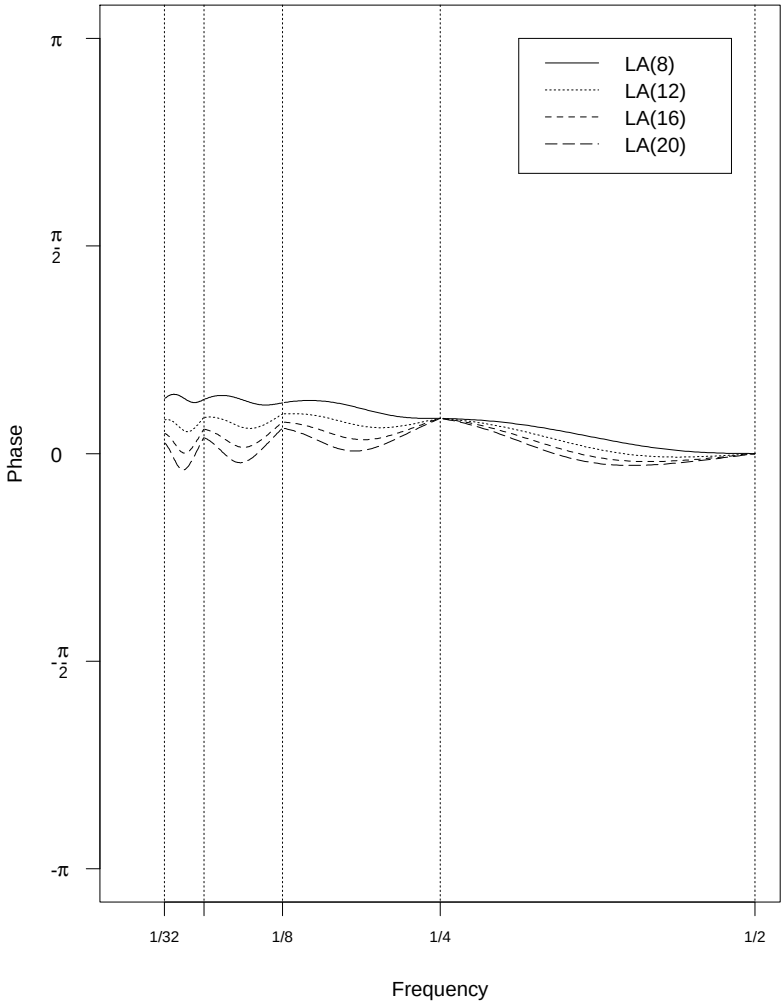

Figure 1.

LIST OF FIGURES

Figure 1 — Phase functions of shifted wavelet filters $\{h_{k,l}\}$ obtained from the least-asymmetric filters $\{g_l\}$ and $\{h_l\}$ of lengths $L = 8, 12, 16, 20$. The phases shown were got by applying the advance $-\nu_k^{(H)}$ to the raw coefficients, where $\nu_k^{(H)}$ is given in (8). Here $L/2$ is even in each case.

Figure 2 — As for Fig. 1, but here for $L = 10, 14, 18$, so that $L/2$ is odd in each case.

Figure 3 — Phase of unshifted Haar wavelet filters.

L	l	coefficients
8	0	+0.045570345896
	1	+0.017824701442
	2	−0.140317624179
	3	−0.421234534204
	4	+1.136658243408
	5	−0.703739068656
	6	−0.041910965125
	7	+0.107148901418
10	0	+0.038654795955
	1	−0.041746864422
	2	−0.055344186117
	3	−0.281990696854
	4	+1.023052966894
	5	−0.896581648380
	6	+0.023478923136
	7	+0.247951362613
	8	−0.029842499869
	9	−0.027632152958
12	0	−0.011031867509
	1	−0.002499922093
	2	+0.063250562660
	3	+0.029783751299
	4	−0.102724969862
	5	−0.477904371333
	6	+1.113892783926
	7	−0.694457972958
	8	−0.068323121587
	9	+0.166863215412
	10	+0.004936612372
	11	−0.021784700327
14	0	+0.003792658534
	1	+0.001481225915
	2	−0.017870431651
	3	−0.043155452582
	4	+0.096014767936
	5	+0.070078291222
	6	+0.024665659489
	7	−0.758162601964
	8	+1.085782709814
	9	−0.408183939725
	10	−0.198056706807
	11	+0.152463871896
	12	+0.005671342686
	13	−0.014521394762
16	0	+0.002672793393
	1	+0.000428394300
	2	−0.021145686528
	3	−0.005386388754
	4	+0.069490465911
	5	+0.038493521263
	6	−0.073462508761
	7	−0.515398670374
	8	+1.099106630537
	9	−0.680745347190
	10	−0.086653615406
	11	+0.202648655286
	12	+0.010758611751
	13	−0.044823623042
	14	−0.000766690896
	15	+0.004783458512
18	0	+0.001981193736
	1	−0.000876502539
	2	−0.018769396836
	3	+0.016303351226
	4	+0.042744433602
	5	−0.000825140929
	6	−0.077172161097
	7	−0.337658923602
	8	+1.015259790832
	9	−0.873048407349
	10	+0.049882830959
	11	+0.270893783503
	12	−0.025786445930
	13	−0.087791251554
	14	+0.012528896242
	15	+0.014515578553
	16	−0.000669141509
	17	−0.001512487309
20	0	−0.000649589896
	1	−0.000080661204
	2	+0.006495728375
	3	+0.001137535314
	4	−0.028786231926
	5	−0.008152816799
	6	+0.070703567550
	7	+0.045240772218
	8	−0.050256540092
	9	−0.542813011213
	10	+1.088251530500
	11	−0.667071338154
	12	−0.100240215031
	13	+0.225558972234
	14	+0.016418869426
	15	−0.064950924579
	16	−0.002072363923
	17	+0.012220642630
	18	+0.000135245020
	19	−0.001089170447

Cliffs, New Jersey: Prentice Hall, 1989.

TABLE 1 (follows)

High-pass filter coefficients for Daubechies' least-asymmetric class. These have been subjected to a left-to-right reversal where appropriate. The coefficients are standardized so that the corresponding low-pass filter coefficients (through the quadrature mirror filter relationship) sum to 2.

so that the type of shift defined by $\nu_k^{(G)}$ satisfies condition (11) and the phase of the shifted scaling filters will be equal at the joins of the octave bands.

VI. CONCLUSIONS

We have derived the appropriate advance to apply to Daubechies' least-asymmetric scaling and wavelet filters at each scale, in order to achieve nearest to zero phase. It depends on whether half the length of each of the original quadrature mirror filters is even or odd. The derived shifts ensure that the phase of the appropriately shifted scaling and wavelet filters is continuous across octave bands.

ACKNOWLEDGEMENT

The work of Emma McCoy was supported through an EPSRC (UK) Grant, and that of Don Percival through an NSF Phase I SBIR Grant. The support is gratefully acknowledged.

REFERENCES

[1] W.H. Press, B.P. Flannery, S.A. Teukolsky and W.T. Vetterling, *Numerical Recipes in C* (2nd edition). Cambridge: Cambridge University Press, 1992.

[2] I. Daubechies, *Ten Lectures on Wavelets,* CBMS-NSF Regional Conf. Series in Appl. Math. Philadelphia, PA: SIAM, 1992.

[3] O. Rioul and P. Duhamel, "Fast algorithms for discrete and continuous wavelet transforms," *IEEE Trans. Info. Th.,* vol. 38, pp. 569-586, 1992.

[4] O. Rioul, "Simple regularity criteria for subdivision schemes," *SIAM J. Math. Analysis,* vol. 23, pp. 1545-1576, 1992.

[5] O. Rioul, "A discrete-time multiresolution theory," *IEEE Trans. Signal Processing*, vol. 41, pp. 2591-2606, 1993.

[6] I. Daubechies, "Orthonormal bases of compactly supported wavelets II. Variations on a theme," *SIAM J. Math. Anal.*, vol. 24, pp. 499-519, 1993.

[7] A.V. Oppenheim and R.W. Schafer, *Discrete-Time Signal Processing.* Englewood

If $\theta_k^{(SH)}(f)$ is to equal $\theta_{k-1}^{(SH)}(f)$ at $f = 1/\lambda_{k+1}$ then

$$\begin{aligned}\frac{2\pi}{\lambda_{k+1}}(\eta_k^{(H)} - \eta_{k-1}^{(H)}) &= \theta^{(H)}(1/2) - \theta^{(H)}(1/4) + \theta^{(G)}(1/4) \\ &= -\frac{\pi}{2}(L-1),\end{aligned}$$

where we have used (3) and the fact that $\theta^{(H)}(1/2) = 0$. The right-hand side is of course derived modulo 2π. Hence,

$$(\eta_k^{(H)} - \eta_{k-1}^{(H)}) = -\lambda_{k-1}(L-1). \tag{10}$$

However, from (8) and the fact that $L_k = (2\lambda_k - 1)(L-1) + 1$, it is also true that

$$(\nu_k^{(H)} - \nu_{k-1}^{(H)}) = -\lambda_{k-1}(L-1),$$

so that the type of shift defined by $\nu_k^{(H)}$ satisfies the necessary and sufficient condition (10).

For the scaling filters we can proceed in a similar way. In this case we can use (6a) to obtain

$$\theta_k^{(G)}(f) - \theta_{k-1}^{(G)}(f) = \theta^{(G)}(\lambda_k f).$$

If we represent the phase of the shifted filters in this case by

$$\begin{aligned}\theta_k^{(SG)}(f) &= \theta_k^{(G)}(f) - 2\pi f \zeta_k^{(G)} \\ \theta_{k-1}^{(SG)}(f) &= \theta_{k-1}^{(G)}(f) - 2\pi f \zeta_{k-1}^{(G)}\end{aligned}$$

where $\zeta_k^{(G)}$ is an arbitrary shift at scale λ_k then we obtain

$$\frac{2\pi}{\lambda_{k+1}}(\zeta_k^{(G)} - \zeta_{k-1}^{(G)}) = \pi,$$

but since ν is always odd we can write this, modulo 2π, as

$$\frac{2\pi}{\lambda_{k+1}}(\zeta_k^{(G)} - \zeta_{k-1}^{(G)}) = \pi\nu,$$

and hence

$$(\zeta_k^{(G)} - \zeta_{k-1}^{(G)}) = \lambda_k \nu. \tag{11}$$

From (7) and the fact that $L_k = (2\lambda_k - 1)(L-1) + 1$, it is also true that

$$(\nu_k^{(G)} - \nu_{k-1}^{(G)}) = \lambda_k \nu,$$

IV. EXAMPLE PHASES OF WAVELET FILTERS AT EACH SCALE

For the wavelet filters the closest to zero-phase which can be obtained is got by applying the advance $-\nu_k^{(H)}$, to the raw coefficients $\{h_{k,l}\}$, where $\nu_k^{(H)}$ is given by (8) for the different lengths L. The resulting exact phase functions after making the shifts are shown in Figs. 1 and 2 for the octave bands $(1/4, 1/2), (1/8, 1/4), (1/16, 1/8), (1/32, 1/16)$ corresponding to $k = 0, ..., 3$. Fig. 1 is for cases when $L/2$ is even, and Fig. 2 when $L/2$ is odd, as in (8). Clearly, a better approximation to zero-phase results when $L/2$ is even.

By way of comparison, Fig. 3 shows the phase of the Haar wavelet filters (with no time shifts). It is linear over the octave bands, making the output of the DWT seriously out-of-phase with the input. Hence, while the Haar system is appealing from the simplicity point of view, we should be aware of problems such as poor octave-band filtering, and phase-shifting.

V. CONTINUITY OF PHASE ACROSS OCTAVE BANDS

An interesting observation from the Figures is that the phase functions of the wavelet filters following their time advance are continuous across the octave bands marked by the vertical dashed lines. While this is obviously desirable, we need to understand under what conditions this happens. From (6b) we note that

$$\theta_k^{(H)}(f) = \theta^{(H)}(\lambda_k f) + \sum_{j=0}^{k-1} \theta^{(G)}(\lambda_j f)$$

$$\theta_{k-1}^{(H)}(f) = \theta^{(H)}(\lambda_{k-1} f) + \sum_{j=0}^{k-2} \theta^{(G)}(\lambda_j f)$$

so that

$$\theta_k^{(H)}(f) - \theta_{k-1}^{(H)}(f) = \theta^{(H)}(\lambda_k f) - \theta^{(H)}(\lambda_{k-1} f) + \theta^{(G)}(\lambda_{k-1} f). \tag{9}$$

We require the phase of the shifted filters to be equal at the joins of the octave bands. Let us represent the phase of the shifted filters by

$$\theta_k^{(SH)}(f) = \theta_k^{(H)}(f) - 2\pi f \eta_k^{(H)}$$

$$\theta_{k-1}^{(SH)}(f) = \theta_{k-1}^{(H)}(f) - 2\pi f \eta_{k-1}^{(H)}$$

where $\eta_k^{(H)}$ is an arbitrary shift at scale λ_k. Hence from (9)

$$\theta_k^{(SH)}(f) + 2\pi f \eta_k^{(H)} - \theta_{k-1}^{(SH)}(f) - 2\pi f \eta_{k-1}^{(H)} = \theta^{(H)}(\lambda_k f) - \theta^{(H)}(\lambda_{k-1} f) + \theta^{(G)}(\lambda_{k-1} f).$$

For reference purposes the high-pass filter coefficients giving (4) are tabulated in Table 1; these are standardized in the same way as Table 6.3 of [2], i.e., so that the corresponding low-pass filter coefficients sum to 2.

We now want to consider the phase functions of the wavelet and scaling filters at higher scales $\lambda_k \equiv 2^k$ for $k \geq 1$. Let $\{g_{k,l}\}$ and $\{h_{k,l}\}$ represent the scaling and wavelet filters for scale λ_k. Let $G_k(\cdot)$ and $H_k(\cdot)$ be the transfer functions corresponding to the filters $\{g_{k,l}\}$ and $\{h_{k,l}\}$, and let $\theta_k^{(G)}(\cdot)$ and $\theta_k^{(H)}(\cdot)$ be the corresponding phase functions. Let us apply Fourier transforms to (2). We obtain

$$G_k(f) = G(2^k f)G_{k-1}(f) = G(\lambda_k f)G_{k-1}(f) \Rightarrow G_k(f) = \prod_{j=0}^{k} G(\lambda_j f) \tag{6a}$$

$$H_k(f) = H(2^k f)G_{k-1}(f) = H(\lambda_k f)G_{k-1}(f) \Rightarrow H_k(f) = H(\lambda_k f)\prod_{j=0}^{k-1} G(\lambda_j f) \tag{6b}$$

Using (4), we thus obtain from (6a) and (6b)

$$\theta_k^{(G)}(f) = \sum_{j=0}^{k} \theta^{(G)}(\lambda_j f) \approx \sum_{j=0}^{k} 2\pi\lambda_j f\nu = 2\pi f(\lambda_{k+1} - 1)\nu = 2\pi f\nu_k^{(G)}$$

and

$$\begin{aligned}\theta_k^{(H)}(f) = \theta^{(H)}(\lambda_k f) + \theta_{k-1}^{(G)}(f) &\approx -2\pi\lambda_k f(L - 1 + \nu) + 2\pi f(\lambda_k - 1)\nu \\ &= -2\pi f(\lambda_k[L-1] + \nu) = 2\pi f\nu_k^{(H)},\end{aligned}$$

where

$$\nu_k^{(G)} \equiv (\lambda_{k+1} - 1)\nu \quad \text{and} \quad \nu_k^{(H)} \equiv -(\lambda_k[L-1] + \nu).$$

Since the length L_k of the filters $\{g_{k,l}\}$ and $\{h_{k,l}\}$ can be expressed as $L_k = (2\lambda_k - 1)(L-1) + 1$, we have from (5),

$$\nu_k^{(G)} = \frac{L_k - 1}{L - 1}\nu = \begin{cases} -\frac{(L_k-1)(L-2)}{2(L-1)}, & \text{if } \frac{L}{2} \text{ is even;} \\ -\frac{(L_k-1)L}{2(L-1)}, & \text{if } L = 10 \text{ or } 18; \\ -\frac{(L_k-1)(L-4)}{2(L-1)}, & \text{if } L = 14; \end{cases} \tag{7}$$

and

$$\nu_k^{(H)} = -\left(\frac{L_k}{2} + \frac{L}{2} + \nu - 1\right) = \begin{cases} -\frac{L_k}{2}, & \text{if } \frac{L}{2} \text{ is even;} \\ -\frac{L_k}{2} + 1, & \text{if } L = 10 \text{ or } 18; \\ -\frac{L_k}{2} - 1, & \text{if } L = 14. \end{cases} \tag{8}$$

The motivation for Daubechies' class of least asymmetric (LA) filters was to design a scaling filter that has, as closely as possible, a linear phase function, i.e.,

$$\theta^{(G)}(f) \approx 2\pi f\tilde{\nu}$$

for some integer $\tilde{\nu}$ (exact linear phase cannot be achieved with compactly supported scaling filters – see Daubechies [2, Theorem 8.1.4]). For the corresponding wavelet filter, this approximation implies that

$$\theta^{(H)}(f) \approx -2\pi f(L-1+\tilde{\nu}) + \pi(\tilde{\nu}-1),$$

so that the wavelet phase function has, approximately, *generalized* linear phase. (Our use of the terms 'linear' and 'generalized linear' phase follows the definitions in [7], namely that generalized linear phase is linear phase plus a phase shift or intercept term.) Hence, the wavelet phase function will simply have linear phase if $\tilde{\nu}$ is odd; this is clearly desirable since then a simple time shift will result in the wavelet phase function being reduced to approximately zero phase. The LA scaling filter coefficients are given in Table 6.3 of Daubechies [2] for even filter lengths L ranging from 8 to 20. Computations indicate that $\tilde{\nu}$ is odd for $L = 8$, 12, 18 and 20, but is even for 10, 14, and 16; however, if the tabulated filter coefficients for the last three cases are subjected to a left-to-right reversal, the transfer function for the reversed scaling filter will be $\exp\left[-i2\pi f(L-1)\right]G^*(f)$ and hence will have a phase function given by

$$-\theta^{(G)}(f) - 2\pi f(L-1) \approx -2\pi f(\tilde{\nu}+L-1) \equiv -2\pi f\nu,$$

where ν is now odd. In what follows, we assume that the scaling filter coefficients for $L = 10$, 14, and 16 are reversed so that (letting $\nu \equiv \tilde{\nu}$ for $L = 8$, 12, 18 and 20) all the LA scaling and wavelet phase functions can be taken to be approximately linear phase:

$$\theta^{(G)}(f) \approx 2\pi f\nu \quad \text{and} \quad \theta^{(H)}(f) \approx -2\pi f(L-1+\nu), \tag{4}$$

where ν is always odd. In particular, computations indicate that

$$\nu = \begin{cases} -\frac{L}{2}+1, & \text{if } \frac{L}{2} \text{ is even;} \\ -\frac{L}{2}, & \text{if } L = 10 \text{ or } 18; \\ -\frac{L}{2}+2, & \text{if } L = 14. \end{cases} \tag{5}$$

which has $L_2 = 2L_1 + L - 2$ coefficients, and it is clear that in general for $k \geq 1$,

$$L_k = 2L_{k-1} + L - 2.$$

Exactly analogous results apply for the scaling filters. For example, the scaling filter $\{g_{1,l}\}$ is got by increasing the sample rate of the low-pass filter $\{g_{0,l}\} \equiv \{g_l\}$ by a factor of 2, followed by low-pass filtering with $\{g_l\}$:

$$\{g_{1,l}\} = (g_{0,0}, 0, g_{0,1}, 0, g_{0,2}, 0, ..., 0, g_{0,L-1}) * (g_0, g_1, g_2, ..., g_{L-1}).$$

and the number of coefficients is the same as for $\{h_{1,l}\}$, etc.

III. PHASE OF SCALING AND WAVELET FILTERS AT EACH SCALE

We consider the Daubechies least-asymmetric wavelet class where a quadrature mirror relationship holds between the basic scaling filter $\{g_l\}$ and the wavelet filter $\{h_l\}$. This relationhip is defined here as [5]:

$$h_l = (-1)^l g_{L-l-1} \quad \text{or} \quad g_l = (-1)^{l+1} h_{L-l-1} \quad \text{for} \quad l = 0, \ldots, L-1.$$

We assume that the filters are applied without an advance; i.e., the quantity $\sum_{l=0}^{L-1} h_l x_{n-l}$ is regarded as the filter output at index n. Let $G(\cdot)$ and $H(\cdot)$ be the transfer functions corresponding to the filters $\{g_l\}$ and $\{h_l\}$, and let

$$G(f) = \exp\left[i\theta^{(G)}(f)\right] |G(f)| \quad \text{and} \quad H(f) = \exp\left[i\theta^{(H)}(f)\right] |H(f)|,$$

where $\theta^{(G)}(\cdot)$ and $\theta^{(H)}(\cdot)$ are the phase functions for the filters. For even L, the quadrature mirror filter relationship gives the result

$$H(f) = \exp\left[-i2\pi f(L-1) + i\pi\right] G\left(\frac{1}{2} - f\right),$$

so that

$$\exp\left[i\theta^{(H)}(f)\right] |H(f)| = \exp\left[-i2\pi f(L-1) + i\pi\right] \exp\left[i\theta^{(G)}\left(\frac{1}{2} - f\right)\right] \left|G\left(\frac{1}{2} - f\right)\right|,$$

giving us the following relationship between the wavelet and scaling phase functions:

$$\theta^{(H)}(f) = \theta^{(G)}\left(\frac{1}{2} - f\right) - 2\pi f(L-1) + \pi. \tag{3}$$

where the discrete scaling filters (sequences) and wavelet filters (sequences) are given by

$$\begin{aligned} g_{k,l} &= \sum_{j \in \mathbf{Z}} g_{k-1,j} g_{l-2j} \, , \\ h_{k,l} &= \sum_{j \in \mathbf{Z}} h_{k-1,j} g_{l-2j} \, , \end{aligned} \tag{1}$$

with $\{g_{0,l}\} \equiv \{g_l\}$ and $\{h_{0,l}\} \equiv \{h_l\}$. Here one goes from scale to scale by applying the interpolation operator

$$f_l \rightarrow \sum_{j \in \mathbf{Z}} f_j g_{l-2j},$$

i.e., increase the sample rate of $\{g_{k-1,j}\}$ or $\{h_{k-1,j}\}$ by a factor of 2, and then apply a half-band low-pass filter (e.g., [3]). An alternative way of generating the discrete scaling and wavelet sequences is to go from scale 1 to scale 2^k by increasing the sample rate of $\{g_j\}$ or $\{h_j\}$ by a factor of 2^k and then apply a 2^{-k}th-band low-pass filter. This gives

$$\begin{aligned} g_{k,l} &= \sum_{j=0}^{L-1} g_{k-1,l-2^k j} g_j \, , \\ h_{k,l} &= \sum_{j=0}^{L-1} g_{k-1,l-2^k j} h_j \, , \end{aligned} \tag{2}$$

(e.g., [4]). The discrete scaling and wavelet filters $\{g_{k,l}\}$ and $\{h_{k,l}\}$ (designed to satisfy a regularity condition), when plotted against $l2^{-k}$, uniformly converge to a scaling function $\phi(t)$ and a wavelet $\psi(t)$ as $k \rightarrow \infty$, [5, p. 2602]. The discrete scaling filters and discrete wavelet filters in (1) and (2) are specialized low-pass and octave band-pass filters, respectively, and in this paper we study their phase properties when the filters $\{g_l\}$ and $\{h_l\}$ are chosen in the Daubechies least-asymmetric class [2, Chapter 8], [6].

Consider the wavelet filter $\{h_{1,l}\}$. Using the approach outlined in (1) we can see it is got by increasing the sample rate of the high-pass filter $\{h_{0,l}\} \equiv \{h_l\}$ by a factor of 2, followed by low-pass filtering with $\{g_l\}$. If $\{g_{0,l}\}$ and $\{h_{0,l}\}$ are causal finite impulse response filters with an even number of coefficients $L_0 = L$ say, then we can write $\{h_{1,l}\}$ as the convolution

$$\{h_{1,l}\} = (h_{0,0}, 0, h_{0,1}, 0, h_{0,2}, 0, ..., 0, h_{0,L-1}) * (g_0, g_1, g_2, ..., g_{L-1}).$$

This means that the number of coefficients of $\{h_{1,l}\}$ is $L_1 = L + L - 1 + L - 1 = 3L - 2$. Similarly we can write $\{h_{2,l}\}$ as the convolution

$$\{h_{2,l}\} = (h_{1,0}, 0, h_{1,1}, 0, h_{1,2}, 0, ..., 0, h_{1,L_1-1}) * (g_0, g_1, g_2, ..., g_{L-1}),$$

larger the index, the smaller the space; this is the same convention used by Daubechies [2]. Each element x_n of the input sequence could be either the inner product of the continuous-time $L^2(\mathbf{R})$ function $x(t)$ with the scaling function $\phi(t-n)$, or, thought of purely in discrete time, it is simply equal to the nth element of the $\ell^2(\mathbf{Z})$ sequence $\{x_n\}$. We call V_0 the approximation space at scale $\lambda_0 = 2^0 = 1$, so that $\{x_n \equiv x_{0,n}\}$ is at scale $2^0 = 1$. An element of V_k is at scale $\lambda_k = 2^k$. Then we can write x_n in terms of a coarse approximation $A_1 x_n \in V_1$, at scale $\lambda_1 = 2$, and a missing detail $D_1 x_n \in W_1$, where $V_0 = V_1 \oplus W_1$:

$$x_n = A_1 x_n + D_1 x_n ,$$

where

$$A_1 x_n = \sum_{j \in \mathbf{Z}} x_{1,j} g_{2j-n} ,$$

with $\{x_{1,j}\}$ got by filtering $\{x_{0,n}\}$ by a real-valued low-pass filter $\{g_l : l = 0, ..., L-1\}$, appropriately chosen, followed by subsampling by 2:

$$x_{1,j} = \sum_{l \in \mathbf{Z}} g_{2j-l} x_{0,l} = \sum_{l=0}^{L-1} g_l x_{0,2j-l} .$$

Similarly,

$$D_1 x_n = \sum_{j \in \mathbf{Z}} d_{1,j} h_{2j-n} ,$$

with $\{d_{1,j}\}$ got by filtering $\{x_{0,n}\}$ by a real-valued high-pass filter $\{h_l : l = 0, ..., L-1\}$, appropriately chosen, followed by subsampling by 2:

$$d_{1,j} = \sum_{l \in \mathbf{Z}} h_{2j-l} x_{0,l} = \sum_{l=0}^{L-1} h_l x_{0,2j-l} .$$

Note that we use $\{g_l\}$ to represent a low-pass filter and $\{h_l\}$ for a high-pass filter, rather than vice versa, since this choice seems more logical. Also, $\{g_l\}$ represents the low-pass filter in the analysis part of the subband filtering representation. We can now represent $A_1 x_n \in V_1$ by the sum of $A_2 x_n \in V_2$, at scale $\lambda_2 = 4$, and $D_2 x_n \in W_2$, where $V_1 = V_2 \oplus W_2$. Continuing in this way we find for $k \geq 1$, scaling coefficients and wavelet coefficients given by

$$x_{k,j} = \sum_{l \in \mathbf{Z}} g_{k-1,2^k j-l} x_{0,l} = \sum_{l \in \mathbf{Z}} g_{k-1,l} x_{0,2^k j-l} ,$$
$$d_{k,j} = \sum_{l \in \mathbf{Z}} h_{k-1,2^k j-l} x_{0,l} = \sum_{l \in \mathbf{Z}} h_{k-1,l} x_{0,2^k j-l} .$$

I. INTRODUCTION

When using the discrete wavelet transform it is important that the filters used at each scale are suitably shifted so that energy in the output sequence is appropriately positioned. For example, the subroutine `pwtset` in the section 13.10 on discrete wavelet transforms in [1] which deals with Daubechies' extremal phase (minimum-phase) wavelet filters allows the centering of the support of the filters at each scale, or the centering of their peaks. Zero phase is nearly always a desirable filter characteristic, and since Daubechies designed a class of low-pass (scaling) filters of even lengths to have approximately linear phase (the so-called 'least-asymmetric' class), the question arises as to what shifts should be applied to the corresponding scaling and wavelet filters at each scale in order for them to best approximate zero-phase filters.

In Section II we give the necessary background to the discrete wavelet transform, and carefully define all the notation used. In particular we show how the scaling and wavelet filters are built up from one scale to the next. In Section III we concentrate our attention on Daubechies' class of least-asymmetric compactly supported wavelets. We derive expressions for the approximate phase of the scaling and wavelet filters at scale 1: all the filters are of even length, but if half the length is odd then the published low-pass filter coefficients must be subjected to a left-to-right reversal in order for the phases to be simple linear. We then derive the approximate phase of the scaling and wavelet filters at higher scales, from which the necessary time advances to be applied to the filters follow immediately. Illustrations are given in Section IV. The conditions under which the phase of appropriately shifted scaling and wavelet filters is continuous across octave bands (i.e., across scales) is investigated in Section V; the shifts derived in this paper give rise to continuous phase functions.

II. BACKGROUND TO SCALING AND WAVELET FILTERS

In order to fix notation, we assume we are interested in the discrete wavelet transform (DWT) of discrete-time square-summable one-dimensional real sequences, i.e., $\ell^2(\mathbf{Z})$ sequences $\{x_n\}$ for which, $\sum_{n=-\infty}^{\infty} x_n^2 < \infty$. Let us identify V_0 with the space $\ell^2(\mathbf{Z})$; the frequency support is $f \in (-1/2, 1/2)$. In terms of a multiresolution analysis, we assume the following nesting of subspaces: $\cdots \subset V_3 \subset V_2 \subset V_1 \subset V_0$. Note that the

Tech. Rept. TR-95-15, Statistics Section, Imperial College.

On The Phase of Least-Asymmetric Scaling and Wavelet Filters

by

E.J. McCoy, D.B. Percival & A.T. Walden

Abstract – When using the discrete wavelet transform it is important that the filters at each scale are suitably shifted so that energy in the output sequence is appropriately positioned. We derive the form of the advance to apply to Daubechies' least-asymmetric scaling and wavelet filters at each scale, in order to obtain nearest to zero phase. At each scale the appropriate advance depends on whether half the length of each of the original quadrature mirror filters is even or odd. The departures from zero phase of the appropriately shifted wavelet filters are illustrated. The conditions under which the phase of appropriately shifted scaling and wavelet filters is continuous across octave bands (i.e., across scales) is investigated; the shifts derived here give rise to continuous phase functions.

E.J. McCoy and A.T. Walden are with the Dept. of Mathematics, Imperial College of Science, Technology and Medicine, 180 Queen's Gate, London SW7 2BZ, U.K.

D.B. Percival is with the Applied Physics Lab, University of Washington HN-10, Seattle, WA 98195, U.S.A. and StatSci Division, Mathsoft, 1700 Westlake Ave. N, Seattle, WA 98109, U.S.A.

Correspondence to Prof. A.T. Walden, Dept. of Mathematics, Imperial College of Science, Technology and Medicine, 180 Queen's Gate, London SW7 2BZ, U.K. Phone (0)171 594 8524, E-mail: a.walden@ic.ac.uk

30/10/1995