Non-smooth Wavelets: Graphing functions unbounded on every interval

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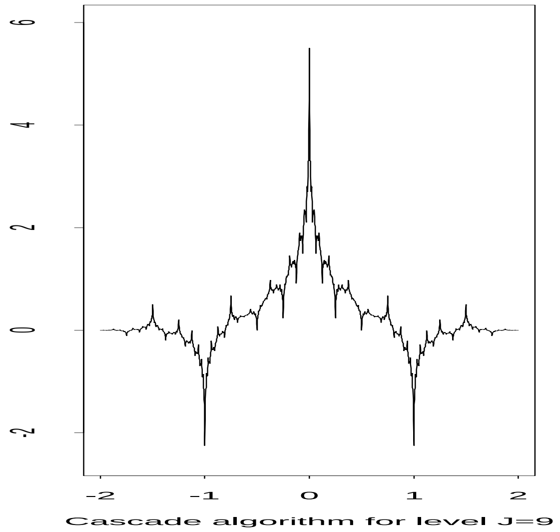
ABSTRACT

Several wavelets from well known biorthogonal families are shown to be unbounded on every interval. One, in fact, is shown to be infinite at each dyadic rational. Not withstanding these facts, we show how to compute exact values for these wavelets at many points and thus achieve exact pictures for these functions.

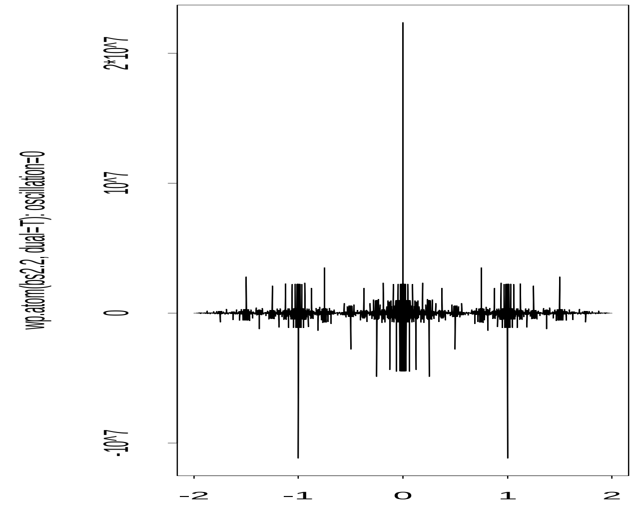
1. Introduction: the origin of our problem.

In the course of developing a Wavelet Toolkit for the S-Plus data analysis and graphics program, the plotting software occasionally produced some strange results for some standard wavelets. One example was the compactly supported scaling function ${}_{2,2}\phi$ of Cohen, Daubechies and Faveau which is biorthogonally dual to the standard linear B-spline scaling function[CDF92, or Chap. 8 in Daub92] . Here is what Daubechies shows[cf. Fig. 8.6 in Daub92] and what was obtained from the plot command `plot(wp.atom("bs2.2",dual=T))`.

CDF plot of ${}_{2,2}\phi(x)$



Toolkit plot of ${}_{2,2}\phi(x)$



Note the wildly different scales on the two plots and yet there is some similarity of "shape". Can these both be "correct"? How can we reconcile the differences? To answer these questions we review the techniques for computing values of a wavelet or scaling function.

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2. Computing wavelet values: dilation equations and the cascade algorithm.

How are values of a wavelet computed? Several approaches have been used all of which ultimately derive from the dilation or duplication equations which provide our only direct knowledge of these functions[Chaps. 6 and 7, Daub92] . Recall that because the scaling (or father) function, ϕ, for a multiresolution analysis (MRA), $V_j \subset L^2(\mathbf{R})$ satisfies

$$\phi(2^j x - n),\ n \in \mathbf{Z},\ \text{form a Riesz basis for } V_j \subset L^2(\mathbf{R}) \tag{1}$$

and the fact that $V_0 \subset V_1$ we derive the existence of a unique l_2 sequence $\{c_n\}$ which is determined from the **dilation (or duplication) equation** for ϕ:

$$\phi(x) = \sum_n c_n \phi(2x - n) \tag{2}$$

Moreover, there are wavelet spaces $W_j \subset V_{j+1}$ (which are particular closed linear complements to V_j) which have a Riesz basis $\psi(2^j x - n)$, $n \in \mathbf{Z}$ for some (mother) wavelet ψ in $W_0 \subset V_1$. Hence there also exists a sequence $\{d_n\}$ in l_2 with

$$\psi(x) = \sum_n d_n \phi(2x - n) \tag{3}$$

Any computation of values for the wavelet ψ is reduced via (3) to computation of ϕ.

We shall restrict our attention to the computationally important case when ϕ is compactly supported so only finitely many c_n differ from 0. Since the compact support of ϕ implies the sum in the wavelet equation (3) is locally finite, all local integrability, smoothness or evaluation properties of the wavelet ψ follow directly from the corresponding properties of the scaling function ϕ. Moreover, the d_n are (usually) rapidly converging to 0, so global properties of ψ also follow from those of ϕ. Because of these facts we shall concentrate solely on deriving computational tools for ϕ.

One of the MRA properties says $\bigcup V_j = L^2(\mathbf{R})$, so the Riesz basis property implies $\int \phi \neq 0$. Then integration of (2) shows that

$$\sum_n c_n = 2. \tag{4}$$

Also an application of the Fourier transform to the dilation equation shows $\hat{\phi}$ satisfies

$$\hat{\phi}(\omega) = m_0(\omega/2)\hat{\phi}(\omega/2) = \prod_{i=1}^{\infty} m_0(\omega/2^i)\hat{\phi}(0)\ , \quad m_0(\omega) = \tfrac{1}{2}\sum c_n e^{in\omega} \tag{5}$$

So the $\{c_n\}$ uniquely determine $\hat{\phi}$ and hence ϕ up to the normalizing constant $\int \phi$ within the class of compactly supported functions.

In addition to the finite support of $\{c_n\}$ we assume that the sum of the integer translates (or periodized version) of ϕ is constant and normalized to be one, i.e.

$$1 = \sum_n \phi(x - n),\ \text{(which is equivalent to } \hat{\phi}(2k\pi) = \delta_0) \tag{6}$$

since then at least the constants are (locally) approximable by sums of translates of ϕ. In this context it is extremely useful to note that the Riesz basis property of the $\phi(x - n)$ imply that $\hat{\phi}((2k+1)\pi) \neq 0$, for some integer k which combined with (5) and (4) implies

$$m_0(\pi) = 0, \quad \text{or} \quad \sum (-1)^n c_n = 0, \quad \text{or} \quad \sum c_{2n} = 1 = \sum c_{2n+1} \tag{7}$$

The **cascade algorithm** for computing approximations to ϕ starts from some (compactly supported) ϕ_0 satisfying (6), such as the characteristic function of some unit interval, i.e. $\chi_{[-1/2,1/2)}$ or $\chi_{[0,1)}$, or the piece-wise linear "hat" function with $h(x) = 1 - |x|, |x| \le 1$, and recursively defines

$$\phi_{j+1}(x) = \sum_n c_n \phi_j(2x - n).$$

If this sequence converges in some suitable sense (say in $L^2(\mathbf{R})$) then the limit satisfies the *dilation equation* and will be compactly supported and so by uniqueness is ϕ . Usually this is used to calculate $\phi_j(\mathbf{Z}/2^j)$ starting with $\phi_0(\mathbf{Z})$; this is employed in CDF. †

The **dyadic dilation algorithm** provides an alternate method, which produces **exact** values for ϕ. Finite support of $\{c_n\}$ and $\{\phi(k), k \in \mathbf{Z}\}$ means ϕ can be computed on the dyadics from (2) as follows:

Solve the eigenvector equation $T_0 \tilde{\phi} = \tilde{\phi}$, $\tilde{\phi} = \phi(\mathbf{Z})$ i.e. $\phi(k) = \sum_n c_n \phi(2k - n)$ where T_0 is a finite rank operator. Normalize $\phi(\mathbf{Z})$ so (6) is satisfied (at 0) and compute $\phi(\mathbf{Z}/2^{j+1})$ from $\phi(\mathbf{Z}/2^j)$ by $\phi(k/2^{j+1}) = \sum_n c_n \phi(k/2^j - n)$

The Wavelet Toolkit used this dyadic dilation approach to the computation of values of ϕ. Pointwise error estimates for this exact dyadic dilation method are a question of numerical accuracy which is (relatively) easy to determine. Local smoothness estimates also can be derived from properties of T_0 and a related operator T_1 which transforms $\phi(\mathbf{Z})$ into $\phi((2\mathbf{Z} + 1)/2)$ as in [MP89, DL91 , DL92, HC94] . Pointwise or global estimates for the cascade algorithm convergence rate are related to approximation order and can be derived using T_0 and T_1. Let us turn to explicating several of these facts via one example.

3. The hat function via dyadic dilation and the cascade algorithm.

For the MRA given by

$$V_j = \{ f \in L^2(\mathbf{R}) \colon f \text{ continuous with } f \text{ linear on each } [k/2^j, (k+1)/2^j) \},$$

the non-zero part of a scaling function is $\phi(x) = 1 - |x|,\ |x| \le 1$. The geometry of the hat function graph shows that the duplication equation for ϕ is

$$\begin{aligned} \phi(x) &= \tfrac{1}{2}\phi(2x-1) + 1\phi(2x) + \tfrac{1}{2}\phi(2x+1) \\ &= c_{-1}\phi(2x-1) + c_0\phi(2x) + c_1\phi(2x+1) \end{aligned}. \tag{8}$$

(We shall discuss one wavelet complement W_0 to V_0 in V_1 in section 5).

We can compute ϕ using only the dilation equation as follows. First let $\bar{\phi}(x) = [\phi(x-1)\ \phi(x)]^T$, $0 \le x < 1$. Then the dilation equation is summarized on the intervals $0 \le x < 1/2$ and $1/2 \le x < 1$ by

$$\bar{\phi}(x) = \begin{bmatrix} \tfrac{1}{2}\phi(2x-1) \\ \tfrac{1}{2}\phi(2x-1) + 1\phi(2x) \end{bmatrix} = T_0 \bar{\phi}(2x) \text{ and } \bar{\phi}(x) = \begin{bmatrix} 1\phi(2x-2) + \tfrac{1}{2}\phi(2x-1) \\ \tfrac{1}{2}\phi(2x-1) \end{bmatrix} = T_1\, \bar{\phi}(2x-1) \tag{9}$$

where the matrices T_0, T_1 are defined by

† In the printing of [Daub92] from which we worked there was no mention of the method for producing Fig. 8.6, p. 273. A revised printing mentions that the cascade algorithm was used.

$$T_0 = \begin{bmatrix} c_{-1} & 0 \\ c_1 & c_0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \text{and } T_1 = \begin{bmatrix} c_0 & c_{-1} \\ 0 & c_1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}.$$

More generally suppose $x = .\,d_1 d_2 d_3 \cdots$ is the dyadic expansion of $x \in [0, 1)$, then the iterated dilation equation is $\bar{\phi}(x) = T_{d_1}\bar{\phi}(\tau(x)) = T_{d_1}T_{d_2}\cdots T_{d_j}\bar{\phi}(\tau^j(x))$, where

$$\tau(x) = 2x - \lfloor 2x \rfloor (= .\,d_2 d_3 \cdots)\text{, and note that } d_j(x) = \left\lfloor 2\tau^{j-1}(x) \right\rfloor \tag{10}$$

In particular, at a dyadic $x = k/2^j$ with $\tau^j(x) = 0$, $\bar{\phi}(x)$ is determined by $\bar{\phi}(0)$ via $\bar{\phi}(x) = T_{d_1}\cdots T_{d_j}\bar{\phi}(0)$ while $\bar{\phi}(0) = [\phi(-1)\ \phi(0)]^T$ is an eigenvector for T_0 for the eigenvalue 1 which satisfies $\phi(-1) + \phi(0) = 1$ (i.e. (6) is satisfied at 0). Since only one such (normalized) eigenvector, $\mathbf{v}$, for T_0 exists, we compute that $\bar{\phi}(0)$ must be $[0\ 1]^T = [.\,0\ 1 - .\,0]^T$. Hence just knowing the c_n and thus the effects of T_0 and T_1 determines ϕ at the dyadics from the iterated dilation equation, i.e. a simple computation yields $\bar{\phi}(.\,d_1 \cdots d_j) = T_{d_1}\cdots T_{d_j}\mathbf{v} = [.\,d_1 \cdots d_j\ \ 1 - .\,d_1 \cdots d_j]^T$. Other calculable values of $\bar{\phi}(x)$ come from x's with a(n) (eventually) periodic dyadic expansion, like $\frac{1}{3} = .\,\overline{01}$, (repeat the 01), since then $\bar{\phi}(1/3) = T_0 T_1 \bar{\phi}(1/3)$. But $\mathbf{v}_{1/3} = [1/3\ 2/3]^T = [.\,\overline{01}\ \ 1 - .\,\overline{01}]^T$ is easily shown to be the unique solution to $T_0 T_1 \mathbf{v} = \mathbf{v}$ with $\sum(\mathbf{v})_i = 1$. So $\bar{\phi}(1/3) = [1/3\ 2/3]^T$ and (2) shows $\bar{\phi}(.\,d_1 d_2 \cdots d_j\overline{01}) = [.\,d_1 \cdots d_j\overline{01}\ \ 1 - .\,d_1 \cdots d_j\overline{01}]^T$. Similar calculations work for any rational $x = p/q$.

We can also derive convergence results for the cascade algorithm. The vector form of the cascade algorithm says choose any vector function $\bar{\phi}_0(x)$ with $\sum(\bar{\phi}_0(x))_i = 1$ and define $\bar{\phi}_{j+1}(x) = T_{d_1(x)}\bar{\phi}_j(\tau(x))$ Then writing $\bar{\phi}_0(x) = [1/2\ \ 1/2]^T + f(x)[1/2 - 1/2]^T$ and using $\bar{\phi}(x) = [x\ \ 1 - x]^T$, we find that $\bar{\phi}(x) - \bar{\phi}_0 = (2x - 1 - f(x))[1/2\ \ -1/2]^T$. But $T_d[1/2\ \ -1/2]^T = \frac{1}{2}[1/2\ \ -1/2]^T$, $d \in \{0,1\}$, so the iterated dilation equation and vector cascade algorithm imply

$$\bar{\phi}(x) - \bar{\phi}_j(x) = T_{d_1}T_{d_2}\cdots T_{d_j}(\bar{\phi}(\tau^j(x)) - \bar{\phi}_0(\tau^j(x))) = (1/2)^j(2\tau^j(x) - 1 - f(\tau^j(x)))\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

Hence if $f(\tau^j(x))$ is $o(2^j)$ or bounded, we conclude $\phi_j(x) \to \phi(x)$. In particular we get convergence (with rates) for all rational x (for which $\tau^j(x)$ is eventually periodic).

4. General formulation of the cascade and dyadic dilation algorithms.

The details of the previous example leads to two recursions which provide a vector form of the cascade and dyadic dilation algorithms. These provide methods to approximate and to calculate exactly the scaling function for a given dilation sequence $\{c_n\}$.

We begin by fixing the start and end of non-zero values in the dilation sequence. So we assume that the support of the sequence $\{c_n\}$ (and ϕ) lie in $[M, M+N]$. More specifically we assume

$$\begin{aligned} c_n &= 0, \quad n < M \quad \text{or} \quad n > M + N \\ \phi(x) &= 0, \quad x < M \quad \text{or} \quad x \geq M + N. \end{aligned} \tag{11}$$

Next we capture all the non-zero behavior of any function ϕ on $[M, M+N)$ via the $\mathbf{R}^N$ vector function

$$\bar{\phi}(x) = \begin{bmatrix} \phi(x+M) \\ \phi(x+M+1) \\ \vdots \\ \phi(x+M+N-1) \end{bmatrix}, \quad 0 \leq x < 1 \tag{12}$$

Then if we define the matrices

$$T_0[i,j] = c_{2i-j},\ T_1[i,j] = c_{2i-j+1},\ M \le i, j < M+N$$

or

$$T_0 = \begin{bmatrix} c_M & 0 & 0 & \cdots & 0 & 0 \\ c_{M+2} & c_{M+1} & c_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & c_{M+N} & c_{M+N-1} \end{bmatrix},\ T_1 = \begin{bmatrix} c_{M+1} & c_M & 0 & \cdots & 0 & 0 \\ c_{M+3} & c_{M+2} & c_{M+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & c_{M+N} \end{bmatrix} \quad (13)$$

we can pull apart the dilation equation into one version for $x - \lfloor x \rfloor < 1/2$ and one for $x - \lfloor x \rfloor \ge 1/2$ just as in the hat function example (9) to get

$$\bar{\phi}(x) = T_0 \bar{\phi}(2x), 0 \le x < 1/2, \bar{\phi}(x) = T_1 \bar{\phi}(2x-1), 1/2 \le x < 1 \quad (14)$$

Moreover if we use $\tau(x)$ and the binary digits $d_j(x)$ from (10) then just as before we have the (generalized) vector form of the dilation equation for $x = .\,d_1 d_2 \cdots d_j \cdots \in [0,1)$:

$$\bar{\phi}(x) = T_{d_1} \bar{\phi}(.\,d_2 \cdots d_j) = T_{d_1} \bar{\phi}(\tau x) = T_{d_1} \cdots T_{d_j} \bar{\phi}(\tau^j(x)) \quad (15)$$

In particular, we have the **Dyadic Dilation Algorithm** for determining $\bar{\phi}(x)$ at the *dyadic rationals*:

i) **First uniquely solve**:

$$\bar{\phi}(0) = T_0 \bar{\phi}(0),\ \sum \phi(i) = 1 \quad \text{(16).initialization}$$

ii) **Then** from $\bar{\phi}(\tau(x))$ for $x = .\,d_1 \cdots d_j = k/2^j$ **compute** :

$$\bar{\phi}(x) = T_{d_1} \bar{\phi}(\tau(x)) = T_{d_1} T_{d_2} \cdots T_{d_j} \bar{\phi}(0) \quad \text{(16).recursion}$$

Note that (7) says the columns of T_0 (and T_1) sum to 1, which implies $[1 \cdots 1]$ is a (left) eigenvector for the eigenvalue 1. Whether there exists a *unique* normalized(right) eigenvector, so (16).initialization can be done, depends on the particular dilation sequence $\{c_n\}$.

The vector recursion form of the **cascade algorithm** for approximating ϕ is:

i) **Choose**

$$\bar{\phi}_0(x) \text{ with } supp(\bar{\phi}_0(x)) \subset [M, M+N) \text{ and } \sum \bar{\phi}_0(x)_i = 1 \quad \text{(17).initialization}$$

ii) **Compute**

$$\bar{\phi}_{j+1}(x) = T_{\lfloor 2x \rfloor} \bar{\phi}_j(\tau x) = T_{d_1(x)} \bar{\phi}_j(\tau(x)) \quad \text{(17).recursion}$$

While most expositions note the use of the *dilation equation* in the form (16) to compute the values of ϕ at the *dyadics*, in fact the value **at any rational** can be computed from (17). For any rational $x = p_1/q \in [0,1)$ has an eventually periodic dyadic expansion $x = .\,d_1 d_2 \cdots d_k \overline{e_1 \cdots e_l}$ since $\tau^{j-1}(x) = p_j/q$ is eventually periodic. Thus we define the **rational dilation algorithm** to compute $\bar{\phi}(p_1/q)$

i) **Find the unique normalized eigenvector solution to:**

$$\bar{\phi}(.\,\overline{e_1 \cdots e_l}) = (T_{e_1} T_{e_2} \cdots T_{e_l}) \bar{\phi}(.\,\overline{e_1 \cdots e_l}),\ \sum \phi(i + .\,\overline{e_1 \cdots e_l}) = 1 \quad \text{(18).initialization}$$

ii) **If** $l \ge 1$ **compute**

$$\bar{\phi}(x) = T_{d_1} \bar{\phi}(\tau(x)) = T_{d_1} T_{d_2} \cdots T_{d_k} \bar{\phi}(.\,\overline{e_1 \cdots e_l}) \quad \text{(18).recursion}$$

Again we should note that the column sums of (each factor of) $T_{e_1} \cdots\cdot T_{e_k}$ are again 1 (i.e. $[1 \cdots 1]$ is a (left) eigenvector with eigenvalue 1), so (right) eigenvectors exist. However, the eigenspace for 1 may intersect the normalized vectors in many, one or no points. Hence the (unique) initialization step may not be possible. If no normalized eigenvectors exist, then the function $\bar{\phi}$ is can not be defined at any of the points $\tau^j(x)$. If there are multiple normalized eigenvectors for $T_e := T_{e_1} \cdots T_{e_k}$ then their difference is an eigenvector for 1 in the space $W = \{\mathbf{w}: \sum \mathbf{w}_i = 0$, the orthogonal complement to $[1 \cdots 1]^T$. Hence multiple initializations exist only if 1 is in the spectrum of $T_e|W = \{\mathbf{w}: \sum \mathbf{w}_i = 0\}$.

5. A biorthogonal dual of the hat function.

We now use this general framework to analyze the behavior of the graphs shown at the start of this paper. In the next section we will extract a few general results from our work.

The CDF function ${}_{22}\phi$ considered in section 1 is the scaling function for an MRA $\tilde{V}_j$ which is biorthogonally dual to the MRA V_j of piecewise linear splines considered in Section 3. What this means is that if we start with the duplication equation for $\tilde{\phi} = {}_{22}\phi$:

$$\begin{aligned}
\tilde{\phi}(x) = {} & \tfrac{1}{4}\big(-1\tilde{\phi}(2x+2) + 2\tilde{\phi}(2x+1) + 6\tilde{\phi}(2x) + 2\tilde{\phi}(2x-1) - 1\tilde{\phi}(2x-2) \big) \\
= {} & \tilde{c}_{-2}\tilde{\phi}(2x+2) + \tilde{c}_1\tilde{\phi}(2x+1) + \tilde{c}_0\tilde{\phi}(2x) + \tilde{c}_1\tilde{\phi}(2x-1) + \tilde{c}_2\tilde{\phi}(2x-2)
\end{aligned}$$

and let ϕ be the hat function (the scaling function for V_0) with dilation sequence $(c_{-1}, c_0, c_{1)} = \frac{1}{2}(1,2,1)$ then the fact that that $\sum_n c_n \tilde{c}_{n+2k} = 2\delta_0(k)$ can be used to establish the *biorthogonality* of $\phi(x-n)$ and $\tilde{\phi}(x-k)$, i.e. $\int \phi(x-n)\tilde{\phi}(x-k)dx = \delta_0(n-k)$. Thus inner products with $\tilde{\phi}(2^j x - k)$ determine the coefficients in the expansion of any $f \in V_j$ with respect to the Riesz basis $\phi(2^j x - n)$. (See [pp. 271-278, Daub92])

At this point we know $\tilde{\phi}(x)$ only through its dilation equation. Let us use the vector algorithms (16) and (18) from Section 4 to get further information about $\tilde{\phi}$. First we exhibit the T_0 and T_1 matrices, since we will want to make several eigenvalue calculations with them:

$$T_0 = \frac{1}{4}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 6 & 2 & -1 & 0 \\ -1 & 2 & 6 & 2 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad T_1 = \frac{1}{4}\begin{bmatrix} 2 & -1 & 0 & 0 \\ 2 & 6 & 2 & -1 \\ 0 & -1 & 2 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Then $\mathbf{v}_1 = [0 \ -1/4 \ 1/2 \ -1/4]^T$ satisfies $T_0\mathbf{v}_1 = \mathbf{v}_1$ and is a basis for the eigenspace for T_0 for the eigenvalue 1. Since $\mathbf{v}_1$ is not normalizable, i.e. $\sum(\mathbf{v}_1)_i = 0$ we have

Fact i)

$$\bar{\tilde{\phi}}(0) \textbf{ is undefined(singular), so for all dyadics } \bar{\tilde{\phi}}(k/2^j) \textbf{ is undefined} \tag{19}$$

What happens in the cascade algorithm? We note that the natural starting point is any function $\bar{\phi}_0(x)$ satisfying (6) at 0. One such choice is the constant function: $\bar{\phi}_0(x) = \mathbf{v}^*{}_1 = [0\ 0\ 1\ 0]^T$, which corresponds to $\phi_0(x) = \chi_{[0,1)}(x)$. Since $T_0\mathbf{v}^*{}_1 = \mathbf{v}^*{}_1 + \mathbf{v}_1$, $\mathbf{v}^*{}_1$ is a generalized eigenvector for T_0 for the eigenvalue 1. Since the eigenvectors $\mathbf{v}_\lambda$ for the remaining eigenvalues of 1/2 and -1/4 satisfy $\sum(\mathbf{v}_\lambda)_i = 0$ any normalized vector, when expanded in the basis $\{\mathbf{v}^*{}_1, \mathbf{v}_1, \mathbf{v}_{1/2}, \mathbf{v}_{-1/4}\}$ will have a coefficient of 1 in the $\mathbf{v}^*{}_1$ direction . We use this to show

Fact ii)

The cascade algorithm for $\tilde{\phi}(x)$ diverges at every dyadic rational!

By the remarks above $(T_0 - I)\bar{\phi}_0(0) = \mathbf{v}_1 + a\mathbf{v}_{1/2} + b\mathbf{v}_{-1/4}$ for any starting $\bar{\phi}_0(0)$ so for $j > l$ if $x = .\, d_1 \cdots d_l$ we have

$$\bar{\phi}_{j+1}(x) - \bar{\phi}_j(x) = T_{d_1} \cdots T_{d_l} T_0^{j-l}(\bar{\phi}_1(0) - \bar{\phi}_0(0)).$$
$$= T_{d_1} \cdots T_{d_l}(\mathbf{v}_1 + a(1/2)^{j-l}\mathbf{v}_{1/2} + b(-1/4)^{j-l}\mathbf{v}_{-1/4})$$

So by addition we deduce that

$$\bar{\phi}_n(.\, d_1 \cdots d_l) = T_{d_1}T_{d_2} \cdots T_{d_l}\bar{\phi}_0(0) + (n-l)T_{d_1}T_{d_2} \cdots T_{d_l}(\mathbf{v}_1)$$
$$+ T_{d_1}T_{d_2} \cdots T_{d_l}(a(1-(1/2)^{n-l+1})/(1/2)\mathbf{v}_{1/2} + b(1-(-1/4)^{n-l+1})/(5/4)\mathbf{v}_{-1/4}).$$

Since the $(n-l)T_{d_1} \cdots T_{d_l}\mathbf{v}_1$ term is not zero its non-zero entries diverge to $\pm\infty$ as $n \to \infty$ and our assertion is verified, as all the other terms are constant or convergent.

Can we say anything about the values of $\bar{\tilde{\phi}}(x)$ and convergence of the cascade algorithm at non-dyadic points? The answer, perhaps surprising, is a resounding yes.

Fact iii)

$\bar{\tilde{\phi}}(x)$ **is finite at every non-dyadic rational.**

Such rationals have the form $p/q = .\, d_1 d_2 \cdots d_l \overline{e_1 \cdots e_k}$ with $0 < \sum e_i < k$. When we consider $T_e = T_{e_1}T_{e_2} \cdots T_{e_k}$, as a maps of $W = \{\mathbf{w} | \sum \mathbf{w}_i = 0\}$ into itself, we can show all its eigenvalues have modulus < 1, i.e the map has spectral radius less than 1, in symbols $\sigma(T_e) = \max\{\ |\lambda| : \lambda \in Sp(T_e|W)\ \} < 1$. Since 1 is an eigenvalue for T_e on all of $\mathbf{R}^4$, any eigenvector $\mathbf{v}_e = T_e\mathbf{v}_e$ is not in W and hence is normalizable. There must be a unique such normalized $\mathbf{v}_e$ since the difference of two distinct such vectors would be an eigenvector in W. This shows $\bar{\tilde{\phi}}(p/q)$ is finite by (18).

Fact iv)

The cascade algorithm converges at every non-dyadic rational

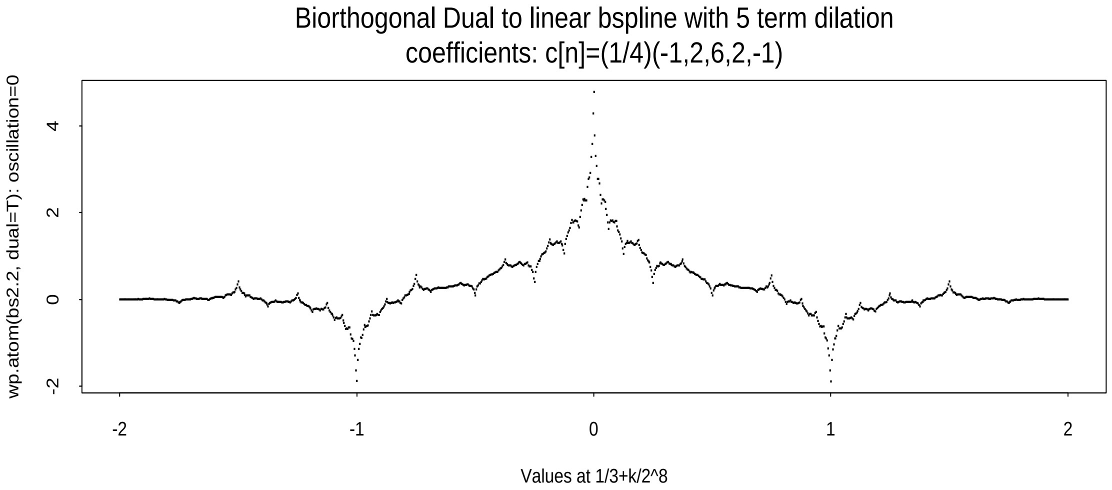
Carrying over the notation in Fact iii), if $x = p/q = .\, d_1 \cdots d_l \overline{e_1 \cdots e_k}$ then $\bar{\tilde{\phi}}(x) = T_{d_1} \cdots T_{d_l}\mathbf{v}_e$ where $\mathbf{v}_e$ is the unique normalized eigenvector for T_e. Write an arbitrary $n = l + jk + m$ with $m < k$. Then $\tau^{n-l}(x) = .\, \overline{e_1 \cdots e_k}$ and $\bar{\tilde{\phi}}_m(.\, \overline{e_1 \cdots e_k} - \mathbf{v}_e) \in W$ as the difference of two normalized vectors. So

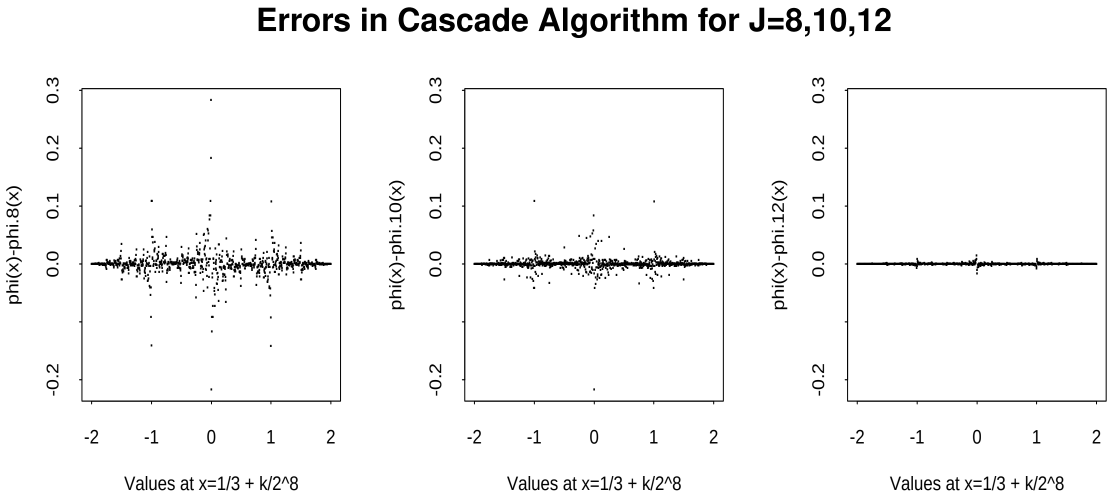
$$\|\bar{\tilde{\phi}}_n(p/q) - \bar{\tilde{\phi}}(p/q)\| = \|T_{d_1} \cdots T_{d_l}(T_e)^j(\bar{\tilde{\phi}}_m(.\, \overline{e_1 \cdots e_m}) - \mathbf{v}_e)\|$$
$$\leq \|T_{d_1} \cdots T_{d_l}\| C_m(\sigma(T_e) + \varepsilon)^j, \text{ any } \varepsilon > 0$$

from the definition of the spectral radius. This goes to 0 as $j \to \infty$ for small ε due to the size estimate for the spectral radius of T_e. This says the cascade algorithm converges at p/q at a rate depending only on the spectral radius of T_e.

These facts provide an explanation for the graphs at the start of this paper. i) and ii) show that the CDF plot is actually quite a bad approximation to the infinite values at the dyadics. On the other hand the Wavelet Toolkit plot is showing a "better" approximation to the infinite values, although one which was "miscomputed", since it was based on normalizing an approximate eigenvector when no normalized eigenvector exists. The normalized approximate eigenvector $\mathbf{v}_0$ had extremely large elements, and its direction was essentially that of $\mathbf{v}_1$. When this vector was used to compute "exact" values via the dyadic dilation method, it actually was as if the cascade algorithm had been started from $\mathbf{v}^*{}_1 + \alpha\mathbf{v}_1$ with α extremely large. Then ii) shows that these values get even larger - since the algorithm diverges to $\pm\infty$.

Finally, iii) and iv) show that we can reinterpret CDF cascade algorithm picture so as to recognize it as an "accurate" approximation. However when the algorithm has been carried out for j stages, so values have been computed at each $x = .\, d_1 \cdots d_j$, then we must interpret the value as an approximation to the value of ϕ at some non-dyadic (rational) point in the interval of length $1/2^j$ with left end point at x. Moreover, the exact values at these non-dyadic (rational) points are given directly by iii). For instance, here is a point plot of the exact values for this ϕ at $1/3 + k/2^j$ for $j = 8$.





Since the spectral radius of $T_e = T_0 T_1$ is easily calculated to be .25 the errors in the cascade algorithm plot at the dyadics as an approximation to this exact plot should be reduced by a factor of 4 for each two iterations. The error plots above confirm this general behavior.

6. Several results and examples concerning singular scaling functions

We extract from the previous examples several results which relate to when scaling functions will fail to be bounded, or have a sequence of values which diverge to $\pm\infty$ at a given rate. All of these generalize what we have learned or will learn from our examples. We shall also give several additional examples illustrating these points.

Proposition

Let c_n be a finite dilation sequence satisfying (4) and (7) with $\phi(x)$ a compactly supported function satisfying (5). Let

$$W = \mathrm{Span}\{\bar{\phi}(y) - \bar{\phi}(z) \colon \bar{\phi}(y),\ \bar{\phi}(z) \in \mathbf{R}^N \text{ and are normalized. }\}$$

and set $x = .\overline{e_1 e_2 \cdots e_k}$, a rational with periodic binary expansion for some (minimal) sequence of binary digits e_i.

i) *If $1 \notin \mathrm{Spec}(T_e|W)$ then T_e has a unique normalized eigenvector $\mathbf{v}_e = T_e \mathbf{v}_e$ and $\bar{\phi}(.\, d_1 \cdots d_l \overline{e_1 \cdots e_k}) = T_{d_1} \cdots T_{d_l} \mathbf{v}_e$ gives the unique definition of $\bar{\phi}$ on the set of $z = m/2^l + x/2^l$.*

ii) *If there exists a normalized $\mathbf{v}_e$ with $(T_e - I)\mathbf{v}_e \neq 0$ but $(T_e - I)^n \mathbf{v}_e = 0$ for some $n > 1$, then $\bar{\phi}$ is undefined at x and the cascade algorithm converges to $\pm\infty$ at each $z = m/2^l + x/2^l \in [0, 1)$.*

iii) *If $\sigma(T_e|W) = \max\{|\lambda| \colon \lambda \mathrm{Spec}(T_e|W)\},1$ and the hypotheses of i) hold, then there exist $y = .\, f_1 \cdots f_n \cdots$ such that for each $z = m/2^l + x/2^l = .\, d_1 \cdots d_l \overline{e_1 \cdots e_k}$*

$$\|\bar{\phi}(z) - \bar{\phi}(.\, d_1 \cdots d_l \overline{e_1 \cdots e_k}^{\,j} f_1 \cdots f_n \cdots)\| \geq C(\sigma(T_e|W) - \varepsilon)^j \text{ as } j \to \infty$$

i.e. there are points $y_j(z)$ with $|z - y_j(z)| < 1/2^{l+kj}$ such that $\|\bar{\phi}(y_j(z))\| \to \infty$ at the rate $O((\sigma(T_e|W) - \varepsilon)^j)$. If the cascade algorithm is started at $\bar{\phi}_0 = \bar{\phi}(y)$ then it diverges at each point z.

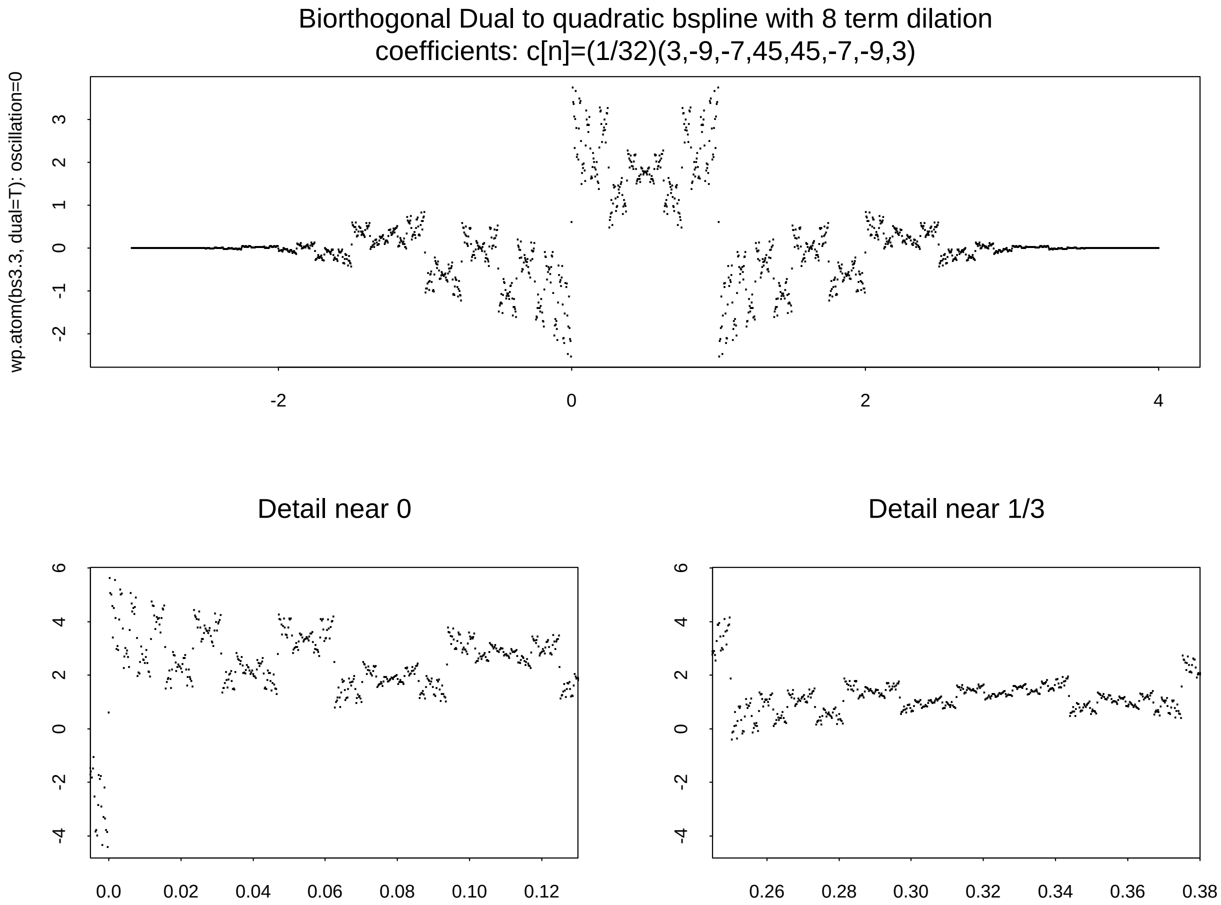
iv) *If $\sigma(T_e|W) < 1$ then the hypotheses of i) must hold and for each $y = .\, f_1 \cdots f_n \cdots$ for which $\bar{\phi}(y)$ is defined and each $z = m/2^l + x/2^l$ with $m/2^l = .\, d_1 \cdots d_l$,*

$$\|\bar{\phi}(.\, d_1 \cdots d_l \overline{e_1 \cdots e_k}^{\,j} f_1 \cdots f_n \cdots) - \bar{\phi}(z))\| \leq C(\sigma(T_e|W) + \varepsilon)^j) \text{ for any } \varepsilon > 0$$

Moreover, the cascade algorithm converges to $\bar{\phi}(z)$

This proposition is related to work in [DL91, DL92, HC94, MP89] but focuses on the divergence behaviors and exploits strongly the full rational dilation equation. A full proof with further results will appear in a subsequent paper.

An interesting example illustrating the divergence/discontinuity behavior of part iii) is provided by the CDF scaling function ${}_{3,3}\phi$ which a compactly supported biorthogonal dual to the quadratic B-spline scaling function with $(c_n \colon n = -1, 0, 1, 2) = \frac{1}{4}(1, 3, 3, 1)$. The dual sequence is $(\tilde{c}_n \colon -3, \cdots, 4) = \frac{1}{32}(3, -9, -7, 45, 45, -7, -9, 3)$. Here the eigenvalue 1 for T_0 has algebraic multiplicity 1 with a normalized eigenvector $\mathbf{v}_1 = (1/400)[0, -3, -41, 244, 244, -41, -3]^T$. Hence the values at the dyadics will be calculable from this unique normalized eigenvector for T_0. However, the maximal eigenvalue for T_0 is 5/4 with (non-normalizable) eigenvector $\mathbf{v}_{5/4} = [0, 1, 15, -50, 50, -15, -1]^T$. Thus the function should be diverging as we approach a dyadic, like 0. On the other hand the maximum eigenvalue for $T_0 T_1$ is 1 and on W the maximum modulus of an eigenvalue is about .5689. Hence the function should appear smoother near $1/3 = .\overline{01}$. Here are some plots which show this behavior:



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