UFTP preprint 357/1994
hep-lat/9402012

Variational description of statistical field theories using Daubechies' wavelets

Christoph Best* and Andreas Schäfer

Institut für Theoretische Physik,
Johann Wolfgang Goethe-Universität,
60054 Frankfurt am Main, Germany

February 16, 1994

Abstract

We investigate the description of statistical field theories using Daubechies' orthonormal compact wavelets on a lattice. A simple variational approach is used to extend mean field theory and make predictions for the fluctuation strengths of wavelet coefficients and thus for the correlation function. The results are compared to Monte Carlo simulations. We find that wavelets provide a reasonable description of critical phenomena with only a small number of variational parameters. This lets us hope for an implementation of the renormalization group in wavelet space.

I Introduction

The success of wavelets [1-6] in analyzing complex signals has prompted speculation about their possible applications to field theories and other lattice systems. Like the Fourier transform, the main feature of the wavelet transform is the separation of scales. It keeps, however, a (coarse-grained) position-space lattice that enables one to mix the features of real- and Fourier-space block-spin transformations, and may thus be a possible candidate for a renormalization group that could be more efficient both in terms of how accurate physical processes are depicted and how much numerical work is necessary to calculate the renormalization group flow. The localization feature of wavelets is especially appealing when one considers theories that have a long-range scale (as the confinement scale in Quantum Chromodynamics). To find out whether a renormalization group description would work in wavelet space, one must learn about how efficiently a field theory can be described in terms of wavelets.

*email: cbest@th.physik.uni-frankfurt.de

In this paper, we study the representation of statistical field theories in wavelet space on the level of the Gaussian approximation. We use a simple variational ansatz to calculate fluctuations in wavelet space. These are related to the correlation function and can be compared to results of Monte Carlo calculations. Other variational approaches based on the Hamiltonian formalism [7] have been quite successful in Quantum Chromodynamics (QCD).

In section II, we give a short introduction to the wavelet transform. Our variational procedure is defined in section III. Its application to the free field theory and to the XY model is demonstrated in section IV. In section V, we apply the method to a Landau-Ginzburg field theory.

II The wavelet transform

II.1 Definition

A wavelet $\psi \in L^2(\mathbf{R})$ is a function whose binary dilatations and dyadic translations generate a Riesz basis of $L^2(\mathbf{R})$. That is, any function in $L^2(\mathbf{R})$ can be expanded into a wavelet series,

$$f(x) = \sum_{n=-\infty}^{\infty} \sum_{x' \in \mathcal{L}^n} \hat{f}^{(n)}(x')\, \psi^{(n)}(x')(x) \quad , \tag{1}$$

where $\psi^{(n)}(x') \in L^2(\mathbf{R})$ denotes the dilatated and translated wavelet, defined by

$$\psi^{(n)}(x')(x) = 2^{-n/2} \psi\left(2^{-n}(x - x')\right) \quad . \tag{2}$$

$n \in \mathbf{Z}$ gives the scale of the wavelet, corresponding to a dilatation by 2^n, and $x' \in \mathcal{L}^n$ the position (translation) on the sublattice $\mathcal{L}^n = 2^n\mathbf{Z}$ of scale n.

The coefficients $\hat{f}^{(n)}(x')$ characterize features of the wavelet at scale n and position x'. In this sense, the wavelet transform offers a mixed position-frequency representation of the signal.

Compact orthonormal wavelets have been constructed by Daubechies [8]. They cannot be given in closed form but by a powerful numerical procedure [9, 10]. The wavelet transform constitutes a *multiresolution analysis* in which the function space $L^2(\mathbf{R})$ is decomposed into a set of nested subspaces

$$\cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \tag{3}$$

generated by binary dilatations of a *scaling function* $\phi(x)$:

$$V_n = \operatorname*{clos}_{L^2(\mathbf{R})} \left\{ \phi^{(n)}(x') : x' \in 2^n\mathbf{Z} \right\} \tag{4}$$

Besides the requirement that their union covers the whole function space and their intersection is the zero function, they have the important property that the contraction

$f(2x)$ of any function $f \in V_n$ is a member of the next finer subspace V_{n+1}. As it turns out, the pair of scaling function and wavelet can be chosen such that the spaces V_n are direct sums of (disjoint) wavelet spaces,

$$W_n = \operatorname*{clos}_{L^2(\mathbf{R})} \left\{ \psi^{(n)}(x') : x' \in 2^n \mathbf{Z} \right\} \quad , \tag{5}$$

$$V_n = V_{n+1} \oplus W_n \quad . \tag{6}$$

Wavelets thus generate a decomposition of the function space into disjoint subspaces W_n, representing different scales. Unlike other methods of multi-resolution analysis, the wavelet spaces and their scaling function spaces are derived from a single pair of functions, $\psi(x)$ and $\phi(x)$, by simple translations and dilatations.

II.2 Construction

The construction of wavelets proceeds from the property that the scaling function $\phi^{(n)} \in V_n$ and the wavelet $\psi^{(n)} \in W_n$ are both members of the next finer subspace V_{n-1}. Thus, they can be expressed as a combination of the basis function $\{\phi^{(n-1)}(x') : x' \in \mathcal{L}^{n-1}\}$ of V_{n-1}:

$$\begin{aligned} \phi^{(n)}(k)(x) &= \sum_l h_l \, \phi^{(n-1)}(k + 2^{n-1} l)(x) \\ \psi^{(n)}(k)(x) &= \sum_l g_l \, \phi^{(n-1)}(k + 2^{n-1} l)(x) \end{aligned} \tag{7}$$

This is the *two-scale relation*. The coefficient series $\{h_l\}_{l \in \mathbf{Z}}$ and $\{g_l\}_{l \in \mathbf{Z}}$ characterize the wavelet. Compactly supported wavelets have a finite number L of nonvanishing coefficients. For orthogonality, the coefficient series must be related by

$$g_l = (-1)^l h_{L-l-1} \quad . \tag{8}$$

Eq. (7) can be inverted to give ($x' \in \mathcal{L}^{n-1} = 2^{n-1}\mathbf{Z}$)

$$\begin{aligned} \phi^{(n-1)}(2x')(x) &= \sum_l \left(h_{2l} \, \phi^{(n)}(2x' - 2^n l)(x) + g_{2l} \, \psi^{(n)}(2x' - 2^n l)(x) \right) \\ \phi^{(n-1)}(2x'+1)(x) &= \sum_l (h_{2l+1} \, \phi^{(n)}(2x' - 2^n l)(x) \\ &\qquad + g_{2l+1} \, \psi^{(n)}(2x' - 2^n l)(x)) \quad . \end{aligned} \tag{9}$$

Eq. (7) is the basis of the partial wavelet transform. If we consider the scalar product of a function $f(x)$ with scaling functions and wavelets, resp.,

$$\bar{f}^{(n)}(x') = \int \phi^{(n)}(x')(x) \, f(x) \, \mathrm{d}x \quad , \tag{10}$$

$$\hat{f}^{(n)}(x') = \int \psi^{(n)}(x')(x) \, f(x) \, \mathrm{d}x \quad , \tag{11}$$

the coefficients $\bar{f}^{(n)}(x')$ give a smoothed representation of $f(x)$, while the wavelet coefficients $\hat{f}^{(n)}(x')$ contain the complementary detail information that is lost in the smoothing process. Using eq. (7), both can be expressed by smoothed coefficients on the next finer scale,

$$\bar{f}^{(n)}(x') = \sum_l h_l \bar{f}^{(n-1)}(x' + 2^{n-1}l) \quad , \tag{12}$$

$$\hat{f}^{(n)}(x') = \sum_l g_l \bar{f}^{(n-1)}(x' + 2^{n-1}l) \quad . \tag{13}$$

Repeating the procedure, any wavelet coefficient can be reduced to the smoothed coefficients at as fine a scale as desired, essentially corresponding to the resolution of the measurement. Using the inverse transformation (9), the basis functions (wavelets and scaling functions) can be reconstructed as accurately as desired.

Eq. (12) defines an orthogonal transformation between the set

$$\{\bar{f}^{(n-1)}(x') : x' \in \mathcal{L}^{n-1}\} \tag{14}$$

of smoothed coefficients at scale $n-1$ and the set of smoothed plus detail coefficients at scale n

$$\{\hat{f}^{(n)}(x'), \bar{f}^{(n)}(x') : x' \in \mathcal{L}^{n-1}\} \quad . \tag{15}$$

If we consider a finite-dimensional vector $(\bar{f}^{(0)}(x'), x' = 0, \ldots, 2^N - 1)$ as input (implying periodic continuation where required), repeated application of this transformation results in the wavelet vector

$$\{\hat{f}^{(n)}(x'), n = 1, \ldots, N, x' \in \mathcal{L}^n; \bar{f}^{(N)}(0)\} \tag{16}$$

As we have only a finite number of elements in the input vector, the partial wavelet transform must stop when there is only one coefficient left, the smoothed coefficient of the coarsest scale, giving the average of $f(x)$. The transformation is orthogonal and thus defines an orthonormal basis, consisting of discretized wavelets, in the space of vectors. The scaling relation (2) applies only approximately to these wavelets. The higher the scale n and the larger the extent of the wavelet, the less is the discretization felt, and the discrete wavelets approach the continuum ones.

Wavelets in more than one dimension can be constructed by direct products. They then carry an additional label $t = 1, \ldots, 2^D - 1$ that specifies their composition. It is advantageous to put both wavelet and scaling function on the same footing using an index $t = 0, 1$,

$$\psi_t^{(1D,n)}(x')(x) = \begin{cases} \phi^{(n)}(x')(x) \quad , & \text{if } t = 0 \\ \psi^{(n)}(x')(x) \quad , & \text{if } t = 1 \end{cases} \tag{17}$$

Then the N-dimensional scaling function ($t = 0$) and wavelets ($t = 1, \ldots, 2^D - 1$) are given by

$$\psi_t^{(n)}(x')(x) = \prod_{k=1}^{D} \psi_{t_k}^{(1D,n)}(x'_k)(x_k), \tag{18}$$

where

$$t = \sum_{k=1}^{D} t_k 2^{k-1} \quad . \tag{19}$$

III Method

III.1 Wavelet representation

We study statistical field theories on the lattice in wavelet representations. The lattice field $a(x)$, $x \in \mathcal{L}$, is expanded over a wavelet basis,

$$a(x) = \sum_{n=1}^{N} \sum_{t}^{n_w} \sum_{x' \in \mathcal{L}^n} \hat{a}_t^{(n)}(x')\, \psi_t^{(n)}(x')(x) \tag{20}$$

$$\hat{a}_t^{(n)}(x') = \sum_{x \in \mathcal{L}^n} \psi_t^{(n)}(x')(x)\, a(x) \quad , \tag{21}$$

on a D-dimensional lattice $\mathcal{L} = \mathcal{L}^0$ with 2^{DN} sites, N giving the number of scales. The sublattices $\mathcal{L}^n$ have accordingly $2^{D(N-n)}$ sites. $t = 0$ or $1, \ldots, n_w = 2^D - 1$ denotes the cartesian composition of multidimensional wavelets. $t = 0$ is included only at the topmost level $n = N$, where $\phi_0^{(N)}$ is the constant function, and $\hat{a}_0^{(n)}$ thus gives the average of $a(x)$ over all sites.

III.2 Variational principle

We employ the principle of minimal free energy to obtain an approximate description of the probability distribution in the partition sum. In a trial ensemble with probability distribution $P(\alpha; \{a(x)\})$ characterized by a set of variational parameters α, we calculate the entropy S

$$S = \int \mathrm{d}\{a(x)\}\, P(\alpha, \{a(x)\}) \ln P(\alpha, \{a(x)\}) \tag{22}$$

and the internal energy U

$$U = \int \mathrm{d}\{a(x)\}\, P(\alpha, \{a(x)\})\, \mathcal{H}(\{a(x)\}) \quad . \tag{23}$$

By minimizing the free energy

$$F = U - \frac{1}{\beta} S \tag{24}$$

with respect to the parameters α, the best-fit probability distribution is obtained.

As our goal is to find out how efficiently wavelets describe statistical field theories, we choose as a trial ensemble the coarsest possibility, a Gaussian uncorrelated ensemble in wavelet space,

$$\langle \hat{a}_{t_1}^{(n_1)}(x_1)\, \hat{a}_{t_2}^{(n_2)}(x_2) \rangle = \delta_{n_1,n_2}\, \delta_{t_1,t_2}\, \delta_{x_1,x_2}\, \mathcal{A}_{t_1}^{(n_1)} \tag{25}$$

with scale-dependent fluctuation strengths $\mathcal{A}_t^{(n)}$. This ensemble makes calculations sufficiently simple and is, at the same time, able to describe nonlocal correlations in position space, as we show below. It can be thought as an extension of the purely local ansatz

$$\langle a(x) a(y) \rangle = \delta_{x,y}\, \mathcal{A} \tag{26}$$

used in the simplest Gaussian approximation, which corresponds in wavelet space to constraining all $\mathcal{A}_t^{(n)}$ to the same value. The introduction of different fluctuation scales in the wavelet transform hence enables us to improve the description of nonlocal fluctuations.

The entropy of this ensemble is easily calculated to be

$$S = \frac{1}{2} \sum_{n=1}^{N} \sum_{t}^{n_w} N_n \ln \mathcal{A}_t^{(n)} \quad + \text{const} \quad . \tag{27}$$

$N_n = |\mathcal{L}^n| = 2^{D(N_0-n)}$ is the number of sites on the n-th sublattice.

III.3 Correlator

Given the fluctuation strengths $\mathcal{A}_t^{(n)}$, the correlator can be calculated by applying the wavelet transform

$$\langle a(x)\, a(y) \rangle = \sum_n \sum_t C_t^{(n)}(x,y)\, \mathcal{A}_t^{(n)} \tag{28}$$

with the elementary wavelet correlator

$$C_t^{(n)}(x,y) = \sum_{x' \in \mathcal{L}^n} \psi_t^{(n)}(x')(x)\, \psi_t^{(n)}(x')(y) \quad . \tag{29}$$

These functions are not translation invariant as the translation group has a complicated representation in wavelet space. Translation invariance of the correlator (28) must thus be restored approximately by the $\mathcal{A}_t^{(n)}$ in the same way as rotational invariance is restored on a lattice correlation function.

The scaling relations between wavelets (which hold approximately for discrete wavelets of sufficiently high scale n) imply

$$C_t^{(n)}(x,y) \approx C_t^{(n)}(x-y) \sim 2^{(n'-n)/2}\, C_t^{(n')}\left(2^{n'-n}(x-y)\right) \quad . \tag{30}$$

The elementary wavelet correlators have finite but nonzero extent (as the wavelets have). Thus, long-range correlations can be obtained by (in wavelet space uncorrelated) fluctuations at sufficiently large scales. In this way, wavelets can map a critical position-space theory with long-range correlations to a less critical wavelet-space theory dominated by short-range correlations.

IV Gaussian and XY model

IV.1 Wavelet fluctuation strengths

The Gaussian model (free field theory) is given by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{x \in \mathcal{L}} \sum_{\mu=1}^{D} \left(a(x+e_\mu) - a(x)\right) + \frac{m^2}{2} \sum_{x \in \mathcal{L}} \left(a(x)^2\right) \quad . \tag{31}$$

It can be solved exactly by a Fourier transform, giving the correlator [11]

$$\langle a(x)a(x+d)\rangle = \frac{1}{\beta}\frac{1}{N_0^2}\sum_{k\in\mathcal{FL}}\frac{\cos(k\cdot d)}{m^2+2\sum_\mu(2-2\cos(k\cdot e_\mu))} \tag{32}$$

with k on the dual lattice $\mathcal{FL}$ with lattice spacing $2\pi/2^N$.

The wavelet correlator can be calculated directly by applying the wavelet transform to this expression:

$$\langle \hat{a}_{t_1}^{(n_1)}(x_1')\, a_{t_2}^{(n_2)}(x_2')\rangle = \sum_{x_1,x_2\in\mathcal{L}} \psi_{t_1}^{(n_1)}(x_1')(x_1)\,\psi_{t_2}^{(n_2)}(x_2')(x_2)\,\langle a(x_1)\,a(x_2)\rangle \quad . \tag{33}$$

On the other hand, fluctuations in wavelet coefficients can be obtained from the variational principle. To this end, we use (25) and calculate the internal energy in the ensemble. To simplify the calculation, we rewrite the next-neighbor difference as a scalar product in lattice field space between the field and the characteristic function of the link,

$$a(x+e_\mu)-a(x) = \langle a, L(x,\mu)\rangle = \sum_{y\in\mathcal{L}} a(y)\,L(x,\mu)(y) \tag{34}$$

with the characteristic function

$$L(x,\mu)(y) = \delta_{y,x+e_\mu} - \delta_{y,x} \quad . \tag{35}$$

As the wavelet transform is orthogonal, it conserves the scalar product

$$a(x+e_\mu)-a(x) = \langle \hat{a}, \hat{L}(x,\mu)\rangle = \sum_{n=1}^{N}\sum_{t}^{n_w}\sum_{x'\in\mathcal{L}^n} a_t^{(n)}(x')\,\hat{L}_t^{(n)}(x,\mu)(x') \tag{36}$$

where $\hat{L}_t^{(n)}(x,\mu)(x')$ is the wavelet transform of $L(x,\mu)(y)$. Then the expectation value of the link Hamiltonian is

$$\left\langle \frac{1}{2}\left(a(x+e_\mu)-a(x)\right)^2\right\rangle_0 = \frac{1}{2}\sum_{n=1}^{N}\sum_{t}^{n_w}|\hat{L}_t^{(n)}(x,\mu)|^2\,\mathcal{A}_t^{(n)} \tag{37}$$

with the symbolic notation

$$|\hat{L}_t^{(n)}(x,\mu)|^2 := \sum_{x'\in\mathcal{L}^n}\left(\hat{L}_t^{(n)}(x,\mu)(x')\right)^2 \tag{38}$$

This gives the power spectrum of the link (x,μ) on scale n, in the sense that the original norm of the function $L(x,\mu)$ is distributed over different scales, subject to conservation of the norm

$$\sum_{n=1}^{N}\sum_{t}^{n_w}|\hat{L}_t^{(n)}(x,\mu)|^2 = \sum_{y\in\mathcal{L}}\left(L(x,\mu)(y)\right)^2 \quad . \tag{39}$$

The notation can be extended to

$$|\hat{L}_t^{(n)}|^2 := \sum_{x\in\mathcal{L}}\sum_{\mu=1}^{D}|\hat{L}_t^{(n)}(x,\mu)|^2 \tag{40}$$

which, so to speak, gives the power spectrum of the whole lattice.

Similarly, the site Hamiltonian gives

$$\frac{m^2}{2} a(x)^2 = \frac{m^2}{2} \sum_{n=1}^{N} \sum_{t}^{n_w} |\hat{S}_t^{(n)}(x)|^2 \, \mathcal{A}_t^{(n)} \tag{41}$$

with the characteristic function $S(x)$ of a site:

$$S(x)(y) = \delta_{y,x} \quad . \tag{42}$$

In this case, there is further analytical simplification as the sum over $x \in \mathcal{L}$ can be performed:

$$\sum_{x \in \mathcal{L}} \left(S_t^{(n)}(x)(x') \right)^2 = \sum_{x \in \mathcal{L}} \left(\psi_t^{(n)}(x')(x) \right)^2 = 1 \tag{43}$$

due to normalization of the wavelets.

The internal energy is therefore

$$U = \langle \mathcal{H} \rangle_0 = \frac{1}{2} \sum_{n=1}^{N} \sum_{t}^{n_w} |\hat{L}_t^{(n)}|^2 \, \mathcal{A}_t^{(n)} + \frac{m^2}{2} \sum_{n=1}^{N} \sum_{t}^{n_w} N_n \, \mathcal{A}_t^{(n)} \quad . \tag{44}$$

Both entropy and internal energy are linear in the variational parameters, so that the variation can be performed analytically, leading to the result

$$\mathcal{A}_t^{(n)} = \frac{1}{\beta} \frac{1}{m^2 + |\hat{L}_t^{(n)}|^2 / N_n} \tag{45}$$

with the notation (40). The quantity $|\hat{L}_t^{(n)}|^2/N_n$ has taken the place of the momentum square k^2 in the corresponding Fourier-space expression.

The wavelet coefficient $n = N, t = 0$ gives the average of the function over all sites. In the case of a characteristic function of a link (35), the average vanishes and thus

$$\mathcal{A}_0^{(N)} = \frac{1}{\beta \, m^2} \tag{46}$$

is divergent for $m^2 \to 0$. The same divergence appears in the Fourier transform associated with the $k = 0$-mode and is caused by the invariance under adding a constant to a massless field. As it appears only in a single wavelet coefficient, it can be as easily subtracted in the wavelet case as with the Fourier transform.

Fig. 1 shows the prediction for the correlator from different wavelets. The higher the order of the wavelet, the more correct the result. This is not surprising as higher wavelets approach more and more the form of free waves that diagonalize the Hamiltonian.

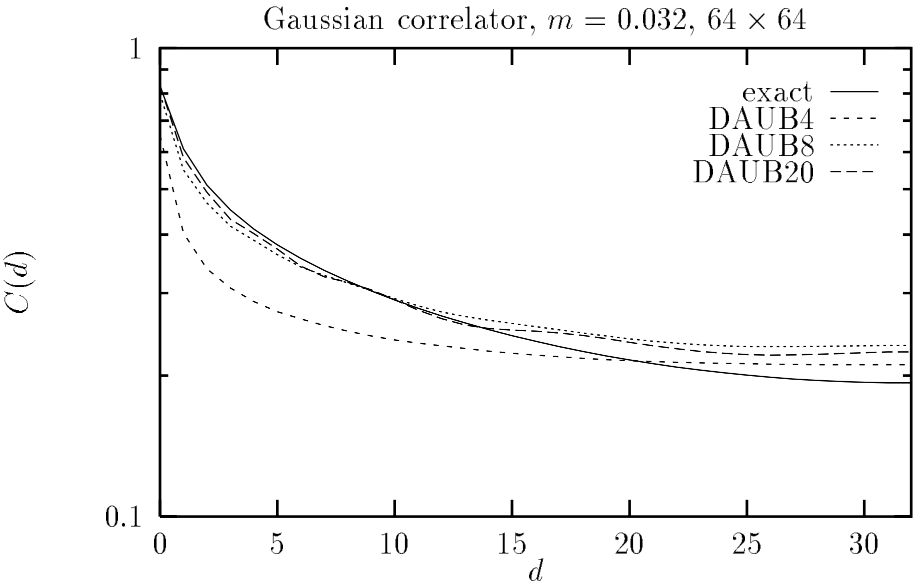

Figure 1: Correlator of a free field at $\beta = 1$. The solid line is the exact result, the broken lines the result of the variational procedure with different wavelet types.

IV.2 Correlator

To calculate the correlator of the massless theory, we first derive the scaling properties of the fluctuation strengths $\mathcal{A}_t^{(n)}$ in respect to n. Using the factorization property (18) of the wavelets, we find for (40)

$$\begin{aligned} \hat{L}_t^{(n)}(x,\mu)(x') &= \psi_t^{(n)}(x + e_\mu - x') - \psi_t^{(n)}(x - x') \\ &= \left(\prod_{k\neq\mu}^{D} \psi_{t_k}^{(1D,n)}(x_k - x'_k) \right) \\ &\quad \times \left(\psi_{t_k}^{(1D,n)}(x_\mu + 1 - x'_\mu) - \psi_{t_k}^{(1D,n)}(x_\mu) \right) \end{aligned} \qquad (47)$$

Summing over x', x, and μ yields

$$|\hat{L}_t^{(n)}|^2 = 2^{D(N-n)} D \sum_x \left(\psi_{t_k}^{(1D,n)}(x_\mu + 1) - \psi_{t_k}^{(1D,n)}(x_\mu) \right)^2 \quad . \qquad (48)$$

The one-dimensional finite difference can be approximated by a derivative as the discrete wavelets approach the continuum wavelets (2) for large n

$$\begin{aligned} &\psi_{t_k}^{(1D,n)}(x_\mu + 1) - \psi_{t_k}^{(1D,n)}(x_\mu) \\ \approx\ & 2^{-n/2} \left(\psi(2^{-n}(x+1)) - \psi(2^{-n}x) \right) \approx 2^{-3n/2} \psi'(2^{-n}x) \quad . \end{aligned} \qquad (49)$$

Consequently

$$\begin{aligned} &\sum_x \left(\psi_{t_k}^{(1D,n)}(x_\mu + 1) - \psi_{t_k}^{(1D,n)}(x_\mu) \right)^2 \\ \approx\ & 2^{-3n} \sum_x \left(\psi'(2^{-n}x) \right)^2 \approx 2^{-2n} \int \mathrm{d}x \ (\psi'(x))^2 \end{aligned} \qquad (50)$$

and we deduce the scaling relation

$$|\hat{L}_t^{(n)}|^2 \sim 2^{-(D+2)n} \qquad \Longrightarrow \qquad \mathcal{A}_t^{(n)} \sim \frac{4^n}{2\beta} \quad . \tag{51}$$

We also note here that, in the massive case $m \neq 0$, (45) implies that the $\mathcal{A}_t^{(n)}$ tend to the constant value $1/\beta m$. As we shall see, this corresponds to the absence of long-range correlations.

For the wavelet autocorrelation $C_t^{(n)}(x, y)$ function, we capture its gross features by

$$C_t^{(0)}(x, x+d) \approx (1 - 2d)\,\Theta(1-d)\,\Theta(d) \quad . \tag{52}$$

The correlation function of the field theory is then

$$\langle a(x)a(x+d)\rangle = \sum_{n=1}^{N}\sum_{t}^{n_w} \mathcal{A}_t^{(n)}(1 - 2^{1-n}d)\,\Theta(1 - 2^{1-n}d)\,\Theta(d) \quad . \tag{53}$$

Let $2^{n_0} < d < 2^{n_0+1}$. Then only terms $n \geq n_0$ contribute in the sum. If $\mathcal{A}_t^{(n)} \sim \alpha^n \mathcal{A}^{(0)}$, the sum can be performed, sending $N \to \infty$,

$$\begin{aligned} C(d) &= \mathcal{A}^{(0)}\, n_w\, \Theta(d) \sum_{n=n_0}^{\infty} \alpha^n\, 2^{-Dn}\,(1 - 2^{1-n}d) \\ &= \mathcal{A}^{(0)}\, n_w\, \Theta(d)\, \alpha^{n_0}\, 2^{-Dn_0} \left(\frac{2^D}{2^D - \alpha} - 2^{1-n_0} d \frac{2^{D+1}}{2^{D+1} - \alpha} \right) \quad . \end{aligned} \tag{54}$$

Putting $2^{n_0} \sim d$ and thus $\alpha^{n_0} \sim d^{\log_2 \alpha}$, we find

$$C(d) \sim d^{-(D-2)} \quad . \tag{55}$$

This is the expected scaling behavior for a Gaussian theory [11]. In two dimensions, it corresponds to a logarithmic correlation function.

IV.3 XY model

We would like to test our approximation on a nontrivial field theory. The two-dimensional XY model is among the simplest field theories with a nontrivial critical behavior.

The Hamiltonian of the XY model is

$$\mathcal{H} = \sum_{x \in \mathcal{L}} \sum_{\mu=1}^{D} \cos\left(a(x+e_\mu) - a(x)\right) \quad . \tag{56}$$

The field $-\pi \leq a(x) < \pi$ gives the angle of the spin at x, relative to some global reference direction. It is symmetric under global $O(1)$ transformations and undergoes a Kosterlitz-Thouless phase transition in two dimensions [11].

As in the Gaussian model, we expand the field $a(x)$ in a wavelet basis, assume Gaussian local fluctuations, and calculate the internal energy as the expectation value of the Hamiltonian in the ensemble. However, to make a Gaussian ansatz, we have to extend the field $a(x)$ to the whole real line. The Gaussian ansatz now does not reflect the periodicity of the Hamiltonian. This is a valid approximation in the ordered low-temperature phase, but it fails at high temperatures. This fact is, however, independent of the wavelet expansion.

To calculate the internal energy, it is useful to note that

$$\langle \cos(x + x_0) \rangle = \exp\left(-\frac{\langle x^2 \rangle}{2} \right) \cos x_0 \tag{57}$$

for a Gaussian variable x. With this and the same procedure as above, we find for the internal energy

$$U = \sum_{x \in \mathcal{L}} \sum_{\mu=1}^{D} \left[1 - \exp\left(-\frac{1}{2} \sum_{n}^{N} \sum_{t}^{n_w} |L_t^{(n)}(x,\mu)|^2 \mathcal{A}_t^{(n)} \right) \right] \quad . \tag{58}$$

In the limit of small fluctuations, the exponential can be expanded, and the (massless) Gaussian model is recovered for the low-temperature limit. Conversely, at high temperatures the exponential vanishes and the internal energy approaches unity per site, reflecting maximal disorder.

The free energy exhibits in this approximation a local minimum that vanishes when the temperature increases. The absence of a global minimum is rooted in the lack of periodicity of the Gaussian probability distribution. While the internal energy U is bounded by unity at maximal disorder, our entropy (27) can increase without bounds. The true entropy, however, is bounded by the available configuration space, $S_{\mathrm{max}} = N_0 \ln 2\pi$. By assuming a Gaussian probability distribution, we have extended the configuration space to the infinite real line.

We can, however, still interpret this result from the point of view of ergodicity in a system with a periodic potential: In the low-temperature phase, the system is localized in a small part of the available configuration space. When the temperature increases, the localized solution becomes unstable at some point, but as our trial probability distribution space does not include periodic distributions, the minimization problem has no solution.

Numerically, we find the temperature of this phase transition at $\beta_{\mathrm{crit}} \approx 0.69$ in two dimensions and $\beta_{\mathrm{crit}} \approx 0.45$ in three dimensions. This is consistent both with the result for the purely local ansatz (26), which gives $\beta_{\mathrm{crit}} = e/2D \approx 0.6769\ldots$ in two dimensions and $0.4530\ldots$ in three dimensions, and with the prediction from an analysis of vortices in two dimensions[11] giving $\beta_{\mathrm{crit}} = 2/\pi \approx 0.6366\ldots$.

Wavelets, however, do not only predict the internal energy but also nonlocal correlations that are not included in the position-space ansatz. Fig. 3 compares the lowest wavelet fluctuation strengths as calculated variationally and as measured in a Monte Carlo simulation. This essentially amounts to comparing correlators, which are not predicted by (26). In the Monte Carlo simulation, we have ignored periodicity (i.e. allowed $a(x)$ to be outside the interval $[-\pi, \pi]$) to model the Gaussian ansatz as closely as possible.

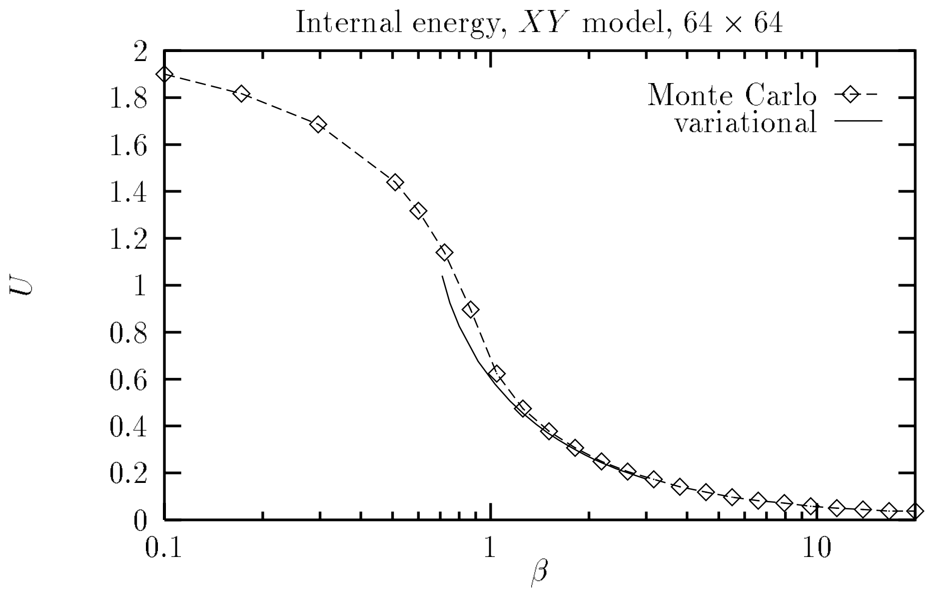

Figure 2: Internal energy per site of the two-dimensional XY model on 64×64 lattice, calculated variationally with wavelets and by a Monte Carlo method.

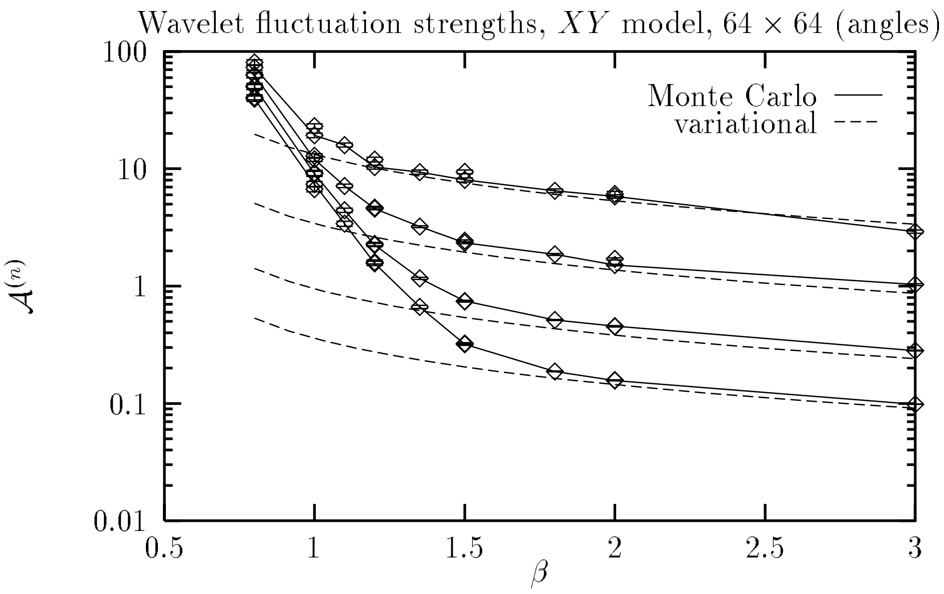

Figure 3: Wavelet coefficient fluctuations in the two-dimensional XY model on a 64×64 lattice.

Above $\beta \approx 1.5$, variational and Monte Carlo result coincide. Below that, nongaussian fluctuations due to periodicity begin to play a role, and the variational method fails. This is also manifest when $\langle [a_t^{(n)}(x')]^4 \rangle$ is measured where a strong departure from Gaussian factorization can be observed.

V Landau-Ginzburg model

In the example of the XY model, major problems were caused by having the (periodic) group parameter (i.e., the angle) as field variable instead of the spin vectors. Since the constraint $s^2 = 1$ on the length of the spin vectors translates to a complicated expression in wavelet space, it is not easily incorporated in the variational ansatz. (The same problem prevented us from considering the even more simple Ising model.)

However, as our main interest will be in renormalization group transformations which dilute the constraints and eventually allow a description in terms of a continuous order parameter governed by an effective theory, we now turn to the study of this effective theory, the Landau-Ginzburg model.

V.1 Wavelet representation

The Landau-Ginzburg model is given by a N_c-component field $a(x)$ with the $O(N_c)$-symmetric Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{x \in \mathcal{L}} \sum_{\mu=1}^{n} (a(x+e_\mu) - a(x)) + \frac{m^2}{2} \sum_{x \in \mathcal{L}} a(x)^2 + \frac{g}{2} \sum_{x \in \mathcal{L}} \left(a(x)^2 \right)^2 \tag{59}$$

corresponding to a $(\Phi^2)^2$ field theory. For $m^2 < 0$ and $g > 0$ it exhibits a "Mexican hat" potential with spontaneous symmetry breaking. In mean-field theory, the phase transition between $O(N_c)$ symmetric and symmetry-broken phase takes place at $\beta = 0$. We wish to estimate the shift of the phase transition temperature due to fluctuations in wavelets.

To allow for symmetry breaking, we must add another variational parameter, the expectation value $\bar{a}$ of the field:

$$\langle a(x) \rangle = \bar{a} \quad . \tag{60}$$

As this is a constant, it implies for the wavelet coefficients

$$\langle \hat{a}_t^{(n)}(x') \rangle = \delta_{n,N} \delta_{t,0} \frac{\bar{a}}{\sqrt{N_0}} \quad , \tag{61}$$

i.e., the only coefficient with a nonzero expectation value is $\hat{a}_0^{(N)}$, associated with the scaling function at the topmost level. The additional variational parameter $\bar{a}$, giving the center of the probability distribution of $\hat{a}_0^{(N)}$, appears only in the internal energy, as a mere shift of the probability distribution does not influence the entropy.

The first two terms of the Hamiltonian (kinetic and mass) have been calculated in the context of the Gaussian model. The only modification is that we now have to deal with a vector-valued field $a_\alpha(x)$, $\alpha = 1, \ldots, N_c$. The wavelet fluctuation strength is thus a tensor

$$\langle \hat{a}^{(n_1)}_{t_1,\alpha_1}(x'_1)\, \hat{a}^{(n_2)}_{t_2,\alpha_2}(x'_2)\rangle = \delta_{n_1,n_2}\delta_{t_1,t_2}\delta_{x'_1,x'_2}\, \mathcal{A}^{(n_1)}_{t_1,\alpha_1\alpha_2} \quad . \tag{62}$$

The unperturbed Hamiltonian appears then as

$$\mathcal{H}_0 = \frac{1}{2}\sum_{n=1}^{N}\sum_{t}^{n_w} |L^{(n)}_t|^2 \;\mathrm{tr}\,\mathcal{A}^{(n)}_t + \frac{m^2}{2}\sum_{n=1}^{N}\sum_{t}^{n_w} N_n \;\mathrm{tr}\,\mathcal{A}^{(n)}_t + \frac{m^2}{2} N_0\,\bar{a}^2 \tag{63}$$

where the trace operator refers to summing over the vector indices α.

To calculate the influence of interactions, we expand the third term of (59) in wavelet space and then can make use of Wick's theorem, as our trial ensemble is Gaussian:

$$\langle abcd\rangle = \langle ab\rangle\,\langle cd\rangle + \langle ac\rangle\,\langle bd\rangle + \langle ad\rangle\,\langle bc\rangle \quad . \tag{64}$$

Then

$$\begin{aligned}
\frac{g}{2}\sum_x\left(\sum_\alpha a_\alpha(x)^2\right)^2 &= \sum_x\sum_{\alpha\beta} a_\alpha(x)a_\alpha(x)a_\beta(x)a_\beta(x) \\
&= \frac{g}{2}\sum_{n_1,n_2}^{N}\sum_{t_1,t_2}^{n_w}\sum_{x'_1,x'_2}\sum_x \left[\psi^{(n_1)}_{t_1}(x'_1)(x)\right]^2\,\left[\psi^{(n_2)}_{t_2}(x'_2)(x)\right]^2 \\
&\qquad\times\left(\mathrm{tr}\,\mathcal{A}^{n_1}_{t_1}\;\mathrm{tr}\,\mathcal{A}^{n_2}_{t_2} + 2\,\mathrm{tr}(\mathcal{A}^{n_1}_{t_1}\,\mathcal{A}^{n_2}_{t_2})\right) \\
&\quad + g\sum_n^N\sum_t^{n_w}\sum_{x'}\sum_x\left[\psi^{(n)}_t(x')(x)\right]^2\,\bar{a}^2\,\left(\mathrm{tr}\,\mathcal{A}^{(n)}_t + 2\mathcal{A}^{(n)}_{t,11}\right) \\
&\quad + \frac{g}{2}N_0\bar{a}^4 \quad .
\end{aligned} \tag{65}$$

We take the symmetry-breaking along the 1-direction in $O(N_c)$ space. The fluctuation matrix at each scale retains the symmetry under the corresponding stabilizer group (i.e. the group of transformations that leave the symmetry breaking vector invariant) and is thus of the form

$$\mathcal{A} = \begin{pmatrix} \mathcal{A} & 0 & 0 & \cdots \\ 0 & \mathcal{A}_\perp & 0 & \cdots \\ 0 & 0 & \mathcal{A}_\perp & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad . \tag{66}$$

$\mathcal{A}_{||}$ gives the fluctuation strength in the direction of the symmetry breaking, $\mathcal{A}_\perp$ perpendicular to it. In particular

$$\begin{aligned}
\mathrm{tr}\,\mathcal{A} &= \mathcal{A}_{||} + (N_c-1)\mathcal{A}_\perp & (67)\\
\mathrm{tr}(\mathcal{A}\mathcal{A}') &= \mathcal{A}_{||}\mathcal{A}'_{||} + (N_c-1)\mathcal{A}_\perp\mathcal{A}'_\perp \quad . & (68)
\end{aligned}$$

With this, the complete internal energy is

$$\begin{aligned}U &= \frac{1}{2}\sum_{n=1}^{N}\sum_{t}^{n_w}\left(|L_t^{(n)}|^2 + m^2 N_n\right)\left(\mathcal{A}_{\|,t}^{(n)} + (N_c-1)\mathcal{A}_{\perp,t}^{(n)}\right)\\ &+ \frac{g}{2}\sum_{n_1 n_2}^{N}\sum_{t_1 t_2}^{n_w} M_{t_1 t_2}^{n_1 n_2}\left[2\mathcal{A}_{\|,t_1}^{(n_1)}\mathcal{A}_{\|,t_2}^{(n_2)} + (N_c^2-1)\mathcal{A}_{\perp,t_1}^{(n_1)}\mathcal{A}_{\perp,t_2}^{(n_2)}\right.\\ &\qquad\left. +2(N_c-1)\mathcal{A}_{\|,t_1}^{(n_1)}\mathcal{A}_{\perp,t_2}^{(n_2)}\right]\\ &+ g\,\bar{a}^2\sum_{n}^{N}\sum_{t}^{n_w} N_n\left(3\mathcal{A}_{\|,t}^{(n)} + (N_c-1)\mathcal{A}_{\perp,t}^{(n)}\right) + \frac{m^2}{2}N_0\bar{a}^2 + \frac{g}{2}N_0\bar{a}^4 \quad . \qquad (69)\end{aligned}$$

The first term gives the original Gaussian Hamiltonian, the second describes interactions between different scales due to the Φ^4 interaction term; the final two terms are due to spontaneous symmetry breaking.

We have introduced the matrix

$$M_{t_1 t_2}^{n_1 n_2} = \sum_{x\in\mathcal{L}}\sum_{x_1'\in\mathcal{L}^1}\sum_{x_2'\in\mathcal{L}^2}\left[\psi_{t_1}^{(n_1)}(x_1')(x)\right]^2\left[\psi_{t_2}^{(n_2)}(x_2')(x)\right]^2 \qquad (70)$$

describing the nonlinear interaction of fluctuations at different scales caused by the quartic term. Due to the factorization properties of wavelets, it can be rewritten as

$$M_{t_1 t_2}^{n_1 n_2} = \prod_{k=1}^{D}\mathcal{M}_{t_{1k} t_{2k}}^{n_1 n_2} \qquad (71)$$

with

$$\mathcal{M}_{t_1 t_2}^{n_1 n_2} = N_{n_1}\sum_{x_2'\in\mathcal{L}^{n_2}}\sum_{x\in\mathcal{L}}\left[\psi_{t_1}^{(1D,n_1)}(x-x_1')\right]^2\left[\psi_{t_2}^{(1D,n_2)}(x-x_2')\right]^2 \qquad (72)$$

derived from one-dimensional wavelets. In particular, the complexity of (69) increases only slightly when higher dimensions are considered.

V.2 Spontaneous symmetry breaking

The value of $\bar{a}$ can be calculated analytically. As it does not enter the entropy, it is determined by the minimum of the internal energy if the $\mathcal{A}_t^{(n)}$ are given. The corresponding condition is

$$\frac{\partial F}{\partial \bar{a}} = m^2 N_0\bar{a} + 2N_0 g\bar{a}^3 + 2g\bar{a}\sum_{n}^{N}\sum_{t}^{n_w} N_n\left(3\mathcal{A}_{\|,t}^{(n)} + (N_c-1)\mathcal{A}_{\perp,t}^{(n)}\right) = 0 \quad . \qquad (73)$$

While $\bar{a} = 0$ is always one solution to this equation, the other one is the root of a quadratic equation,

$$\bar{a} = \sqrt{-\frac{m^2}{2g} - \sum_{n}^{N}\sum_{t}^{n_w} N_n\left(3\mathcal{A}_{\|,t}^{(n)} + (N_c-1)\mathcal{A}_{\perp,t}^{(n)}\right)} \quad . \qquad (74)$$

Without the fluctuations, this is just the mean field result which exhibits spontaneous symmetry breaking for all values of β, provided that $m^2/2g < 0$. The more fluctuations appear, the more $\bar{a}$ is reduced, until the expression under the root becomes negative, and $\bar{a} = 0$ becomes the only solution, corresponding to a symmetric phase.

The value of $\bar{a}^2$ interacts with fluctuations due to the third term in (69). As the fluctuations enter linearly and indiscriminately of their scale, this modifies the effective mass of the theory. Due to the prefactor 3 in the fluctuation sum, it is different for parallel and for perpendicular fluctuations. Inserting the mean-field value of $\bar{a}$ for comparison, we find

$$m \longrightarrow \begin{cases} m_{\|} = m + 6g\bar{a}^2 \approx -2m \\ m_{\perp} = m + 2g\bar{a}^2 \approx 0 \end{cases} \tag{75}$$

The value $-2m$ is the correct Gaussian curvature of the potential for fluctuations against the valley. Fluctuations perpendicular to the magnetization are massless Goldstone modes.

V.3 Numerical solution

We tested the accuracy of this model both for the case of a small anharmonic perturbation and the full phase transition. Eq. (69) was minimized numerically using a combination of the simplex method and simulated annealing [9].

For comparison, a simple Metropolis simulation of the model was performed. The wavelet transform was calculated on statistically independent configurations and the fluctuation strength of wavelet coefficients measured in this ensemble.

One should keep in mind that the trial probability distribution used here makes at least two bold approximations: It neglects all correlations between wavelet fluctuations, and it assumes that all fluctuations are Gaussian. We therefore should expect no more than a gross representation of the model. However, this should be sufficient for a wavelet renormalization group to work, as the renormalization transformations should enhance the physically important features.

V.3.1 Small anharmonic perturbations

Small values of $g > 0$ represent anharmonicities that manifest themselves in the departures from the free-field result, comparable to perturbation theory. Fig. 4 shows the wavelet coefficient fluctuations calculated variationally and compared to the result of a Monte Carlo calculation. Even where the departure from the free-field result is large (at small β), the results are close.

V.3.2 Landau-Ginzburg phase transition

For $g > 0$, $m < 0$, the potential has the “Mexican hat” potential leading to spontaneous symmetry breaking in mean field theory. Fluctuations change the temperature of the symmetry-breaking phase transition from the mean-field value $\beta = 0$ to some finite

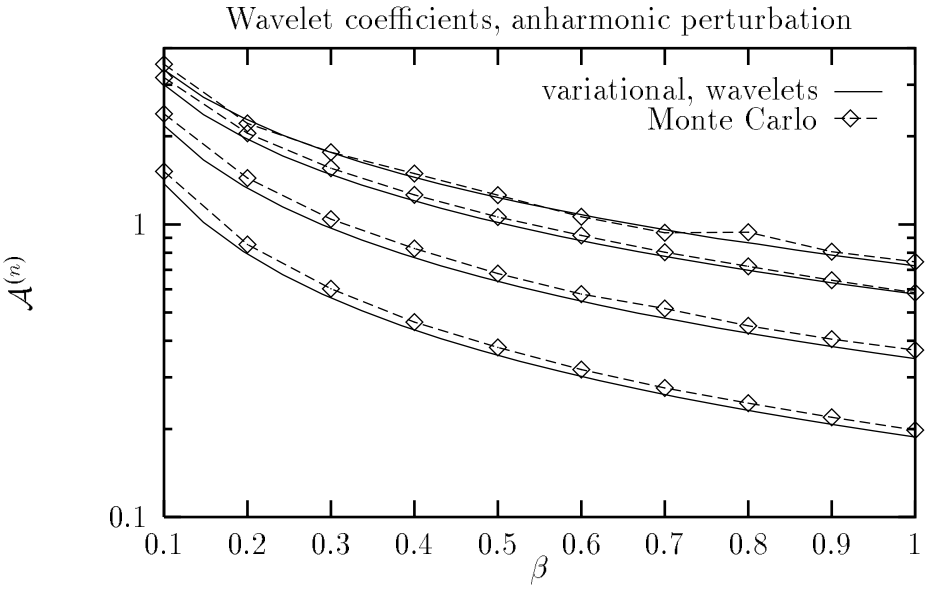

Figure 4: Wavelet coefficient fluctuations in the Gaussian model with an anharmonicity ($m^2 = 1$, $g = 0.2$).

value. In Fig. 5, we have calculated the magnetization variationally both using the wavelet basis and the localized ansatz (26) and by a Monte Carlo calculation on a 32×32 lattice with parameters $m^2 = -4$ and $g = 2$. In the Monte Carlo calculation, an external field $h = 0.05$ was applied. The finite lattice size makes the phase transition less pronounced, but the inclusion of nonlocal fluctuations in the wavelet ansatz lowers, as expected, the phase transition temperature towards the result of the Monte Carlo calculation.

Fig. 6 shows the wavelet fluctuations calculated variationally and in the Monte Carlo simulation. The variational wavelet ansatz gives a faithful representation of the overall features. In particular, the wavelet coefficients both in the Monte Carlo and the variational calculation develop an approximate power scaling near the phase transition, indicating the presence of fluctuations at all scales. In contrary, fluctuations are strong but without scaling in the disordered phase where only short-range correlations persist. In the low-temperature phase, the scaling is that of the massive Gaussian model.

VI Conclusions

We have shown how wavelets can be used to describe fluctuations in a statistical field theory. Using a rather small number of variational parameters, an approximate description of fluctuations in the Gaussian, the XY, and the Landau-Ginzburg model has been obtained. The wavelet coefficients compactly express the scaling features of the field theory in only a few numbers. Comparison to Monte Carlo results leads us to expect that wavelets are a sufficiently efficient basis to merit further investigation of

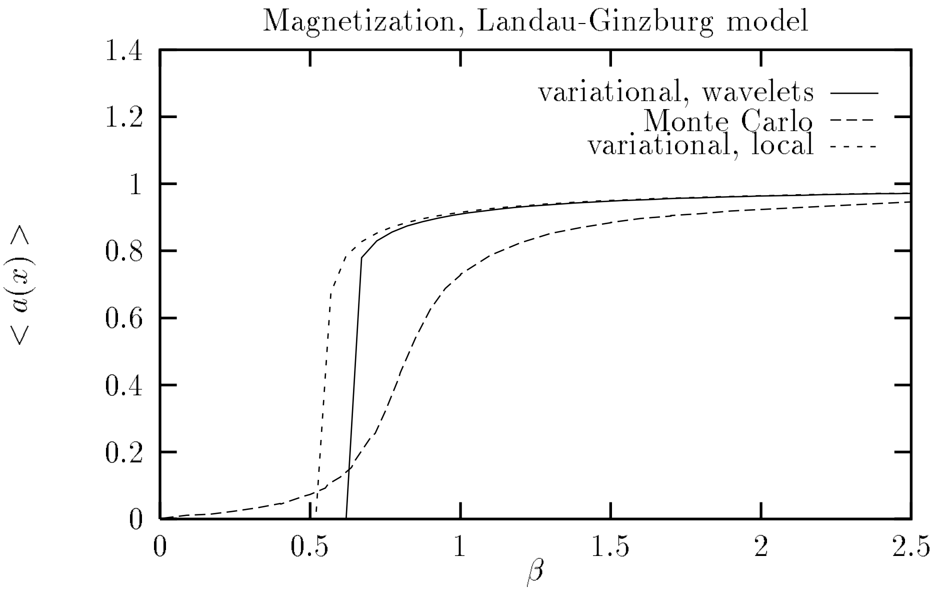

Figure 5: Magnetization in the Landau-Ginzburg model on a 32×32 lattice with $m^2 = -4$, $g = 2$, calculated both in a Monte Carlo simulation and variationally using either wavelets and completely local fluctuations.

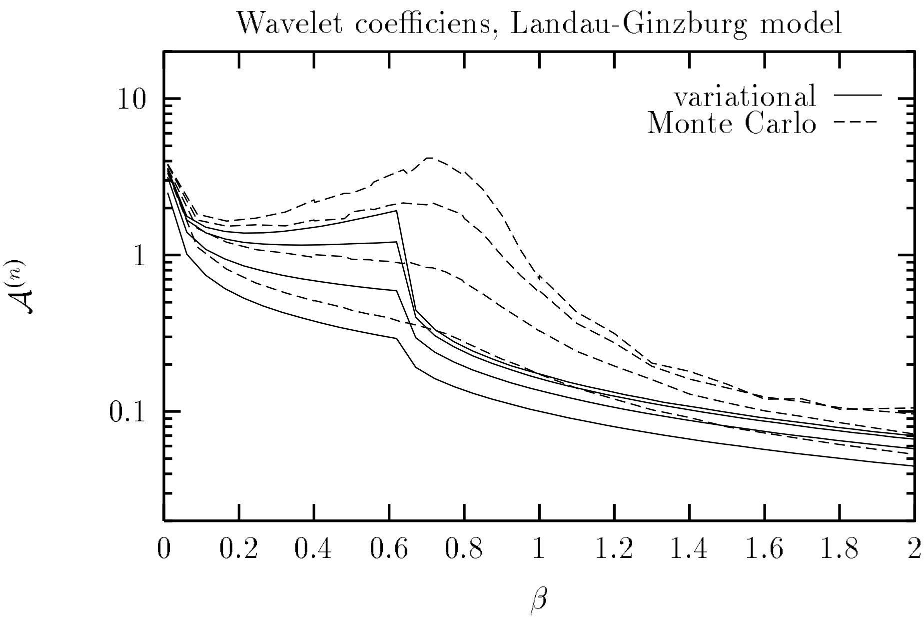

Figure 6: Wavelet coefficient fluctuations in the Landau-Ginzburg model, calculated variationally and in a Monte Carlo simulation.

their use in renormalization group transformations.

As our results are only semiquantitative, we have refrained from calculating critical exponents directly. Since the results are arrived at numerically, it is very difficult to extract critical exponents by differentiating thermodynamic quantities. Once an implementation of the renormalization group in wavelet space is in place, they can be extracted much easier.

Such renormalization group transformations can be implemented by considering the wavelet coefficients as the basic dynamical variables of the system and then integrating them out scale by scale. The wavelet transform has the advantage of having a hierarchy built-in, unlike lattices where the hierarchy is imposed *ad hoc*. The renormalization group transformation can be made either in (wavelet) perturbation theory, using a variational principle, or directly when restricting the model to a hierarchical model [12]. Work in these directions is in progress.

We would like to thank T. Biró, P. Carruthers, M. Greiner, and G. Savvidy for fruitful discussions during the progress of this project, and the German National Scholarship Foundation for its support.

References

[1] C. K. Chui, *An Introduction to Wavelets*, Academic Press, Boston, 1992.

[2] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.

[3] Y. Meyer, *Wavelets and Operators*, Cambridge University Press, New York, 1993; *Wavelets: Algorithms and Applications*, SIAM, Philadelphia 1993.

[4] M. B. Ruskai et al., *Wavelets and Their Applications*, Jones and Bartlett, Boston 1992.

[5] M. Greiner, P. Lipa, P. Carruthers, *Wavelet correlations in the p-model*, preprint HEPHY-PUB 586/93, submitted to Phys. Rev. D.

[6] C. Best, A. Schäfer, W. Greiner, *Wavelets as a variational basis for the XY model*, contribution to Lattice 93, Dallas, TX, Oct. 1993 (bulletin board `hep-lat/9311031`).

[7] T. S. Biró, Ann. Phys. (N.Y.) **191**, 1 (1989);
T. S. Biró, Phys. Lett. B**278**, 15 (1992).

[8] I. Daubechies, Comm. Pure and Appl. Math. 41 (1988) 909.

[9] W. H. Press, B. P. Flannery, S. A. Teukolsky, W. T. Vetterling, Numerical Recipes, 2nd ed., Cambridge University Press, 1992.

[10] G. Beylkin, R. Coifman, V. Rokhlin, Comm. Pure and Appl. Math. 44 (1991) 141.

[11] C. Itzykson, J.-M. Drouffe, *Statistical Field Theory*, Cambridge University Press, Cambridge 1989.

[12] G. K. Savvidy, K. G. Savvidy, A. Schäfer, W. Greiner, *Systems with continuous symmetry on the Bethe lattice*, UFTP preprint 320/1992.