ATOMIC DECOMPOSITION OF FUNCTION SPACES AND FRACTIONAL INTEGRAL AND DIFFERENTIAL OPERATORS

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Abstract

The method ot atomic decomposition of Besov and Lizorkin-Triebel function spaces is combined with basic ideas from the theory of singular integral operators and applied to the study of fractional integral and differential operators (FIDO), of which the Riesz potential is considered in detail as a model example. The main new results are: characterization of the Riesz potential by an infinite-dimensional matrix with certain specific properties; generalization of the "lifting" property of the Riesz potential in Besov spaces, for the case of different metric indices; generalization of Sobolev's theorem about boundedness of the Riesz potential between Lebesgue spaces, for the case of Lizorkin-Triebel spaces. Our results can be extended to other FIDO, e.g., the Bessel potential, the Riemann-Liouville, Caputo and Grünwald-Letnikov FIDO, etc. Other possible applications of the atomic decomposition of function spaces via (bi)-orthonormal wavelets to the theory of FIDO are also discussed in brief.

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Key Words and Phrases: smooth atom, biorthonormal wavelet, Besov space, Lizorkin–Triebel space, real interpolation space, complex interpolation space, fractional integral operator, fractional differential operator, Riesz potential

Resumé

La méthode de décomposition atomique des espaces de Bésov et de Lizorkin–Triebel est ici combinée avec des idées de la théorie des opérateurs intégraux singuliers, et est appliquée à l'étude des opérateurs intégraux et différentiels fractionnaires (OIDF), dont le potential de Riesz est considéré comme exemple modèle. Les résultats noveaux principaux sont: le caractérisation du potential de Riesz par une matrice infinie spéciale, généralisation de la propriété "lifting" du potential de Riesz dans les espaces Besoviens, pour le cas d'indices métriques differents; généralisation du théorème de Sobolev de l'action bornée du potential de Riesz entre des espaces de Lebesgue, pour le cas d'espaces de Lizorkin–Triebel. Une extension de nos résultats peut aussi être elaborée, par exemple, pour le potential de Bessel, OIDF de Riemann–Liouville, Caputo et Grünwald–Letnikov, etc. On discute en bref aussi d'autres applications possibles à la théorie des OIDF de la décomposition atomique des espaces fonctionnels par ondelettes (bi)orthonormales.

1. Introduction

In the late 1980-s and the early 1990-s the theory of Besov and Lizorkin-Triebel spaces was linked with the theory of singular integral operators, as developed by the A. P. Calderon school, by the construction of atomic decompositions of these spaces, notably, by Feichtinger and Gröchenig [8], Frazier and Jawerth [9], and Sickel [14]. This new link between the two theories opens a new, so far little explored, general way to study the properties of fractional integral and differential operators (FIDO). Moreover, since FIDO enjoy the additional property of being convolution operators, the relevant results available for general singular integral operators can be brought to a somehow higher level of precision in the case of FIDO.

Both the homogeneous and inhomogeneous version of the function spaces in consideration can be treated in a similar way. Here our discussion will be for the homogeneous case which allows a more concentrated exposition of the underlying ideas. A typical representative of FIDO acting boundedly between homogeneous Besov or Lizorkin-Triebel spaces is the Riesz potential, and we shall consider it in detail as a model example.

Starting from multi-resolution analyses yielding sufficiently regular orthonormal (ot biorthonormal) compactly supported wavelet bases of $L_2(\mathbb{R}^n)$ which are Riesz unconditional bases in Besov and Lizorkin-Triebel spaces for a broad range of the parameters of these space scales, we study the action of FIDO on the elements of the basis. (This approach can be extended to more general (non-biorthogonal) atomic decompositions, available for a broader range of the parameters of the space scale.) We show that if the function space is taken with its equivalent quasi-norm induced by an atomic decomposition, then FIDO can be represented via infinite-dimensional matrices whose entries decay in a specific way away from the main diagonal.

It is a well-known fact in the theory of "horizontal" interpolation (fixed metric, varying smoothness) of Besov spaces that some types of FIDO have the so-called "lifting" property. We derive a generalization of this property for the "diagonal" case (when both the smoothness and the metric indices of the Besov space are allowed to vary).

Another classical result, with relevance to "vertical" interpolation (fixed smoothness, varying metric) and vertical embedding results for Lizorkin-Triebel spaces, is Sobolev's theorem about boundedness of the Riesz potential between Lebesgue spaces. We generalize Sobolev's theorem for the "diagonal" case (when both the smoothness and metric indices of the Lizorkin-Triebel space are variable).

Finally, we consider possible extensions of our present results (about the Riesz potential in homogeneous spaces) to other types of FIDO and function spaces. We also discuss in brief other possible applications of the atomic-decomposition approach to the study of FIDO.

2. Atomic decomposition of Besov and Lizorkin-Triebel spaces

We adopt the same notation as in Bergh and Löfström [1] and Triebel [16]: $\dot{B}^s_{pq}(\mathbb{R}^n)$ and $\dot{F}^s_{pq}(\mathbb{R}^n)$ for the homogeneous Besov and Lizorkin-Triebel spaces over $\mathbb{R}^n$, respectively; $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ for their inhomogeneous analogues. Here $s \in \mathbb{R}$ is the smoothness index and $p: \ 0 < p \leq \infty$, $q: \ 0 < q \leq \infty$ are the two metric indices of the space scale. For definitions and properties of these spaces we refer to the sources named above. Here we only remind the well-known fact that these two space scales contain as partial cases many classical function spaces encountered in approximation theory and mathematical physics (to name few: the homogeneous/inhomoheneous Sobolev spaces $\dot{W}_p^k = \dot{F}_{p2}^k$, $W_p^k = F_{p2}^k$, $1 < p < \infty$, $k \in \mathbb{N}$; Riesz potential spaces $\dot{H}_p^s = \dot{F}_{p2}^s$, Bessel potential spaces $H_p^s = F_{p2}^s$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$; Lipschitz spaces Lip $s = B_{\infty\infty}^s$, and many others). These spaces have many equivalent definitions which reflect the ever increasing range of their applications. For example, one classical definition of Besov spaces is via integral moduli of smoothness; the respective analogue with Lizorkin-Triebel spaces is via averaged finite differences. Another elegant approach is Peetre's: via compactly supported infinitely smooth functions in the Fourier domain. One relatively recent approach (see [8,9,14]) is the so called atomic decomposition of Besov and Lizorkin-Triebel spaces. Essentially, this approach consists in the following: starting from the Calderon-Zygmund decomposition of $\mathbb{R}^n$ (see, e.g., [1]) into hypercubes Q_{jk}, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, with side 2^{-j}, an explicit linearly independent system of smooth compactly supported functions ("atoms") is designed, so that it serves as unconditional basis in Besov and Lizorkin-Triebel spaces, simultaneously for a broad range of p, q, s (which range depends on the concrete choice of the atoms). In brief, the regular distribution $a_{jk}(x)$, $x \in \mathbb{R}^n$, is called a smooth atom corresponding to the hypercube Q_{jk}, if:

(i) $a(x)$ is compactly supported at $3Q_{jk}$;

(ii) $a(x)$ is continuously differentable at least up to order $|\gamma| \leq ([s]+1)_+$ (where γ is a multiindex, $[s]$ is the Gaussian bracket of s and $t_+ = \max(0,t)$) and, moreover, $|D^\gamma a(x)|$, for $|\gamma| \leq ([s]+1)_+$, satisfies a certain upper bound depending on s, p, $|\gamma|$ and n,

(iii) $\int_{\mathbb{R}^n} x^\gamma a(x) dx = 0$, if $|\gamma| \leq N$, where N depends on p, q, s and n.

The Riesz unconditional bases of smooth atoms have the remarkable property of generating equivalent quasi-seminorms in the homogeneous Besov and Lizorkin-Triebel spaces:

$$||f||_{\dot{B}_{pq}^s(\mathbb{R}^n)} \asymp \left\{ \sum_{j=-\infty}^{\infty} [2^{j[s+\lambda(n,p)]} (\sum_{k\in\mathbb{Z}^n} |\langle f, a_{jk}\rangle|^p)^{\frac{1}{p}}]^q \right\}^{\frac{1}{q}}, \tag{1}$$

$$||f||_{\dot{F}_{pq}^s(\mathbb{R}^n)} \asymp \left\| \left[\sum_{j=-\infty}^{\infty} (2^{j[s+\mu(n,p)]} \sum_{k\in\mathbb{Z}^n} |\langle f, a_{jk}\rangle a_{jk}(\cdot)| \,)^q \right]^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n)}, \tag{2}$$

under some mild restrictions on p, q and s, depending on the choice of the atoms. (Here "$\asymp$" denotes the usual equivalence between quasi-norms.) The upper bound in (ii), the values of N in (iii), $\lambda(n,p)$ in (1) and $\mu(n,p)$ in (2) depend on the notation and normalization of the atoms adopted by the different authors and will not be discussed here. Analogous equivalent quasi-norms are available for $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$, too. For further details and modifications we refer to [8],[9],[14]. Here we only note that one important advantage of Sickel's approach, compared to the more general, but somehow less explicit considerations of Frazier and Jawerth, is that in Sickel's construction the atoms are always obtained by dilations and translations of a single function ψ. This result of Sickel has been additionally improved by a recent observation in orthonormal and biorthonormal wavelet theory (see, e.g., Daubechies [6] or Dahmen [5]) that Sickel's function ψ can be chosen so that all atoms gain the additional property of being mutually (bi)orthonormal. Here we give the final product of this evolution, in the orthonormal case. Let ϕ be a scaling function (father wavelet), and let ψ be the corresponding wavelet (mother wavelet), obtained by multi-resolution analysis (see, e.g., [5],[6]) so that for any $j_0 \in \mathbb{Z}$ the functions $\phi_{j_0k_0}(x_1) = 2^{j_0/2}\phi(2^{j_0}x_1 - k_0)$, $\psi_{jk}(x_1) = 2^{j/2}\psi(2^j x_1 - k)$, $x_1 \in \mathbb{R}$, $j = j_0, j_0+1, \dots$, $k_0 \in \mathbb{Z}$, $k \in \mathbb{Z}$, form an orthonormal basis of $L_2(\mathbb{R})$ and $\int_{-\infty}^{\infty} x_1^\lambda \psi(x_1)dx_1 = 0$ holds for all $\lambda \in \mathbb{N} \cup \{0\} : \lambda < r$, for some, henceforward fixed, $r > 0$. Let ϕ be compactly supported in $\mathbb{R}$ (which implies the same for ψ). Assume that $\phi, \psi \in B^r_{\infty\infty}(\mathbb{R})$. Then, since ϕ and ψ are compactly supported, $\phi, \psi \in B^r_{p\infty}(\mathbb{R})$ holds, for every $p: \ 0 < p \le \infty$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, consider

$$\begin{aligned}
&\phi^{[0]}_{0k}(x) = \phi_{0k_1}(x_1)\phi_{0k_2}(x_2)\phi_{0k_3}(x_3)\dots\phi_{0k_n}(x_n),\\
&\psi^{[1]}_{jk}(x) = \psi_{jk_1}(x_1)\phi_{jk_2}(x_2)\phi_{jk_3}(x_3)\dots\phi_{jk_n}(x_n),\\
&\psi^{[2]}_{jk}(x) = \phi_{jk_1}(x_1)\psi_{jk_2}(x_2)\phi_{jk_3}(x_3)\dots\phi_{jk_n}(x_n),\\
&\dots\\
&\psi^{[2^n-1]}_{jk}(x) = \psi_{jk_1}(x_1)\psi_{jk_2}(x_2)\psi_{jk_3}(x_3)\dots\psi_{jk_n}(x_n).
\end{aligned}$$

Denote $\phi^{[0]} = \phi^{[0]}_{00}$, $\psi^{[l]} = \psi^{[l]}_{00}$, $l = 1, \dots, 2^n - 1$. Then, $\phi^{[0]}$, $\psi^{[l]} \in B^r_{p\infty}(\mathbb{R}^n)$ for any $p: \ 0 < p \le \infty$, where $\psi^{[l]}$ is orthogonal to all polynomials of total degree less than r. Besides, $\{\phi^{[0]}_{0k}, \psi^{[l]}_{jk}\}_{j\in\mathbb{Z},\ k\in\mathbb{Z}^n,\ l=1,\dots,2^n-1}$ is an orthonormal basis of $L_2(\mathbb{R}^n)$. Moreover, for $f \in B^s_{pq}(\mathbb{R}^n)$, $0 < p \le \infty$, $0 < q \le \infty$, $n(\frac{1}{p} - 1)_+ < s < r$,

$$f(x) = \sum_{k\in\mathbb{Z}^n} \alpha_{0k}\phi^{[0]}_{0k}(x) + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1} \beta^{[l]}_{jk}\phi^{[l]}_{jk}(x), \ \text{a.e. } x \in \mathbb{R}^n, \tag{3}$$

holds, where $\alpha_{0k} = \langle\phi^{[0]}_{0k}, f\rangle = \int_{\mathbb{R}^n} \phi^{[0]}_{0k}(x)\overline{f(x)}dx$, $\beta^{[l]}_{jk} = \langle\psi^{[l]}_{jk}, f\rangle$, $\overline{z}$ is the conjugate of $z \in \mathbb{C}$. Convergence in (3) is in the quasi-norm topology of the inhomogeneous Besov space $B^s_{pq}(\mathbb{R}^n)$ and, in view of the lower constraint about s, also in every Lebesgue point of f, i.e., almost everywhere on $\mathbb{R}^n$. Furthermore, $B^s_{pq}(\mathbb{R}^n)$ admits the following equivalent quasi-norm:

$$||f||_{B^s_{pq}(\mathbb{R}^n)} \asymp \left\{ \left(\sum_{k\in\mathbb{Z}^n} |\alpha_{0k}|^p \right)^{\frac{q}{p}} + \sum_{j=0}^{\infty} \left[2^{j[s+n(\frac{1}{2}-\frac{1}{p})]} \left(\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1} |\beta^{[l]}_{jk}|^p \right)^{\frac{1}{p}} \right]^q \right\}^{\frac{1}{q}}. \tag{4}$$

For the homogeneous space $\dot{B}^s_{pq}(\mathbb{R}^n)$ it can be shown that, analogously,

$$f(x) = \sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1} \beta^{[l]}_{jk}\psi^{[l]}_{jk}(x), \ \text{a.e. } x \in \mathbb{R}^n, \tag{5}$$

holds modulo polynomials of total degree less than r, and

$$||f||_{\dot{B}^s_{pq}(\mathbb{R}^n)} \asymp \left\{ \sum_{j=-\infty}^{\infty} \left[2^{j[s+n(\frac{1}{2}-\frac{1}{p})]} \left(\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1} |\beta^{[l]}_{jk}|^p \right)^{\frac{1}{p}} \right]^q \right\}^{\frac{1}{q}}. \tag{6}$$

As far as Lizorkin-Triebel spaces are concerned, the expansions (3) and (5) hold also for $f \in F^s_{pq}(\mathbb{R}^n)$ and $f \in \dot{F}^s_{pq}(\mathbb{R}^n)$, respectively, and there are corresponding counterparts of (4) and (6).

Thus, linear operators which are bounded when acting between certain Besov or Lizorkin-Triebel spaces are reduced to infinite matrices of operators acting between weighted $l_q\,(l_p)$-sequence spaces (for Besov spaces), ot weighted $L_p\,(l_q)$-Lebesgue spaces (for Lizorkin-Triebel spaces). Moreover, the infinite matrix $(c^{[l\lambda]}_{jk,j_1æ})$ of any such operator is almost diagonal (i.e., the matrix entries tend quickly to zero away from the main diagonal). This allows the development of fast and simple algorithms for approximate solution of operator equations (involving singular integral operators, compact perturbations of pseudodifferential operators, or differential operators with constant or variable (possibly non-smooth) coefficients). A fast wavelet transform has also been developed. Another important topic, in the context of both Besov and Lizorkin-Triebel spaces is studying the boundedness of linear operators between two spaces of the same space scale, hence, the study of regularity of solutions of the above-named operator equations. The atomic decomposition approach here is based on exploring the images of atoms under the action of the operator in consideration. The image of an atom is sometimes called a molecule (see [9]). In the partial case when the atom is a wavelet, the molecule is called vaguelette (see [11]). The importance of the concept of a molecule is largely due to the fact that the size of the molecule's derivatives and its behaviour at infinity is very informative about the boundedness properties of the operator. This approach is readily applicable to FIDO (which are typically regular or singular integral operators). Being of convolution type, they commute with translations, and this allows obtaining essentially more precise results than in the case of general singular integral operators and differential operators with variable coefficients. In the next section we shall illustrate this by testing the atomic decomposition approach on a model example: the Riesz potential in homogeneous Besov and Lizorkin-Triebel spaces.

3. Main results

Let $S^{'}(\mathbb{R}^n)$ denote the L. Schwartz space of tempered distributions. We define the Riesz potential as

$$I^\alpha f = F^{-1}(|.|^{-\alpha} Ff),\ f \in S^{'}(\mathbb{R}^n),\ \alpha \in \mathbb{R},$$

where F and F^{-1} are the direct and inverse Fourier transforms, with their usual definition on $S^{'}(\mathbb{R}^n)$.

The following are two classical results about the Riesz potential.

THEOREM A. *(Lifting property for I^α in homogeneous Besov spaces, see e.g. [1],[16]) For any $p:\ 1 \le p \le \infty$, $q:\ 1 \le q \le \infty$, $s \in \mathbb{R}$, $\alpha \in \mathbb{R}$, there exist constants $C_\nu:\ 0 < C_\nu < \infty$, $\nu = 1, 2$, such that*

$$||I^\alpha f||_{\dot{B}^{s+\alpha}_{pq}(\mathbb{R}^n)} \le C_1 ||f||_{\dot{B}^s_{pq}(\mathbb{R}^n)},\ \textit{for any}\ f \in \dot{B}^s_{pq}(\mathbb{R}^n),$$

$$||I^{-\alpha} f||_{\dot{B}^{s}_{pq}(\mathbb{R}^n)} \le C_2 ||f||_{\dot{B}^{s+\alpha}_{pq}(\mathbb{R}^n)},\ \textit{for any}\ f \in \dot{B}^{s+\alpha}_{pq}(\mathbb{R}^n),$$

i.e., $||I^\alpha f||_{\dot{B}^{s+\alpha}_{pq}(\mathbb{R}^n)} \asymp ||f||_{\dot{B}^s_{pq}(\mathbb{R}^n)}$.

THEOREM B. *(Sobolev's theorem, see e.g. [13]) Let $1 \le p \le \infty$, $1 \le q \le \infty$, $\alpha > 0$. Then, $I^\alpha:\ L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$ (i.e., I^α is bounded between $L_p(\mathbb{R}^n)$ and $L_q(\mathbb{R}^n)$) if and only if $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

Before generalizing these two results, let us characterize the matrix $(c^{[l\lambda]}_{jk,j_1æ})$, $j, j_1 \in \mathbb{Z}$, $k, æ \in \mathbb{Z}^n$, $l, \lambda = 1, \ldots, 2^n - 1$, corresponding to I^α with respect to the wavelet basis given in (5) and (6), that is, the matrix with entries

$$c^{[l\lambda]}_{jk,j_1æ} = \langle I^\alpha \psi^{[l]}_{jk}, \psi^{[\lambda]}_{j_1æ}\rangle. \tag{7}$$

The first main result is given by the following theorem.

THEOREM 1. *Let $(c^{[l\lambda]}_{jk,j_1æ})$ be the matrix of I^α, $-r < \alpha < r$, for the fixed wavelet basis, and let $d^{[l\lambda]}_{jk,j_1æ}$ be the coefficients of the inverse of $(c^{[l\lambda]}_{jk,j_1æ})$. Then,*

$$\text{(i)} \qquad c^{[l\lambda]}_{jk,j_1æ} = 2^{-\alpha\min(j,j_1)}\langle I^\alpha \psi^{[l+]}, \psi^{[\lambda-]}_{|j-j_1|,k_{\max(j,j_1)}-2^{|j-j_1|}k_{\min(j,j_1)}}\rangle, \tag{8}$$

$$d^{[l\lambda]}_{jk,j_1æ} = 2^{+\alpha\min(j,j_1)}\langle I^{-\alpha} \psi^{[l+]}, \psi^{[\lambda-]}_{|j-j_1|,k_{\max(j,j_1)}-2^{|j-j_1|}k_{\min(j,j_1)}}\rangle, \tag{9}$$

where $k_{\max(j,j_1)} = k$, $l_+ = l$ if $j \geq j_1$, $k_{\max(j,j_1)} = æ$, $l_+ = \lambda$ if $j < j_1$, and $k_{\min(j,j_1)}$, λ_- have the respective alternative values;

(ii) *there exists a constant $C: \ 0 < C < \infty$, depending only on the wavelet ψ, such that*

$$|c^{[l\lambda]}_{jk,j_1æ}| \leq C.2^{-\alpha\min(j,j_1)-|j-j_1|(r+\alpha+n/2)}, \tag{10}$$

$$|d^{[l\lambda]}_{jk,j_1æ}| \leq C.2^{+\alpha\min(j,j_1)-|j-j_1|(r-\alpha+n/2)}, \tag{11}$$

$$\begin{aligned} &\max\left(\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1}|c^{[l\lambda]}_{jk,j_1æ}|^p, \sum_{æ\in\mathbb{Z}^n}\sum_{\lambda=1}^{2^n-1}|c^{[l\lambda]}_{jk,j_1æ}|^p\right)^{\frac{1}{p}} \\ &\qquad \leq C.2^{-\alpha\min(j,j_1)-|j-j_1|[r+\alpha+n(\frac{1}{2}-\frac{1}{p})]}, \end{aligned} \tag{12}$$

$$\begin{aligned} &\max\left(\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1}|d^{[l\lambda]}_{jk,j_1æ}|^p, \sum_{æ\in\mathbb{Z}^n}\sum_{\lambda=1}^{2^n-1}|d^{[l\lambda]}_{jk,j_1æ}|^p\right)^{\frac{1}{p}} \\ &\qquad \leq C.2^{+\alpha\min(j,j_1)-|j-j_1|[r-\alpha+n(\frac{1}{2}-\frac{1}{p})]}, \end{aligned} \tag{13}$$

where $p: \ 0 < p < \infty$, $|\alpha| < r$.

P r o o f. First of all, let us show that $(c^{[l\lambda]}_{jk,j_1æ})$ is invertible. Consider the canonical isomorphism $\pi: \ \dot{B}^s_{pq}(\mathbb{R}^n)/P_r \to l_q(l_p(w_s))$ between the factor space of $\dot{B}^s_{pq}(\mathbb{R}^n)$ modulo polynomials of total degree less than r and the weighted sequence space with quasi-norm defined in the right-hand side of (6). This isomorphism is defined by

$$f \mapsto \pi f = \left\{\langle \psi^{[l]}_{jk}, f\rangle : \ j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \ldots, 2^n - 1\right\}.$$

By (5) and (6), its inverse is

$$\pi^{-1}\beta = \pi^{-1}\left\{\beta^{[l]}_{jk} : \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^n, \ l = 1, \ldots, 2^n - 1\right\} = \sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{l=1}^{2^n-1}\beta^{[l]}_{jk}\psi^{[l]}_{jk}(x).$$

Identifying, as usual for fixed basis, the matrix $(c^{[l\lambda]}_{jk,j_1æ})$ with the linear operator represented by it, we obtain

$$(c^{[l\lambda]}_{jk,j_1æ}) = \pi I^\alpha \pi^{-1} \tag{14}$$

and, since $I^{-\alpha}$ is a unique two-sided inverse of I^{α}, and π^{-1} is a unique two-sided inverse of π, it can now be seen that $(c_{jk,j_1æ}^{[l\lambda]})$ has a unique two-sided inverse

$$(c_{jk,j_1æ}^{[l\lambda]})^{-1} = \pi I^{-\alpha}\pi^{-1}. \tag{15}$$

Denoting the respective entries of $(c_{jk,j_1æ}^{[l\lambda]})^{-1}$ by $d_{jk,j_1æ}^{[l\lambda]}$, and comparing (15) with (14), we obtain from (7) that

$$d_{jk,j_1æ}^{[l\lambda]} = \langle I^{-\alpha}\psi_{jk}^{[l]}, \psi_{j_1æ}^{[\lambda]}\rangle \tag{16}$$

holds, and, therefore, (9), (11) and (13) follow from (8), (10) and (12), respectively, in view of the symmetric range for α.

Let us prove (8). Denote $g_1 = I^{\alpha}\psi_{jk}^{[l]}$, $g_2 = \psi_{j_1æ}^{[\lambda]}$. By definition of $\psi_{j_1æ}^{[\lambda]}$, $g_2 \in L_2(\mathbb{R}^n)$. Since $\psi^{[l]} \in B_{2\infty}^{r}(\mathbb{R}^n)$, it is easy to see that also $\psi_{jk}^{[l]} \in B_{2\infty}^{r}(\mathbb{R}^n)$ holds for every $j \in \mathbb{Z}$, $k \in \mathbb{Z}$. By $-\alpha < r$, $B_{2\rho}^{-\alpha}(\mathbb{R}^n) \hookleftarrow B_{2\infty}^{r}(\mathbb{R}^n)$ holds, for any $\rho :\ 0 < \rho \le \infty$, where $A \hookleftarrow B$ (or $B \hookrightarrow A$) denotes, as usual, continuous embedding of B in A ($B \subset A$ and $||.||_A \le ||.||_B$ for some $c < \infty$). Take $\rho = 2$. Hence, $\psi_{jk}^{[l]} \in B_{22}^{-\alpha}(\mathbb{R}^n)$. Therefore, since $\psi_{jk}^{[l]}$ is orthogonal to any n-variate polynomial of total degree less than r, also $\psi_{jk}^{[l]} \in \dot{B}_{22}^{-\alpha}(\mathbb{R}^n)$ holds (for $\alpha < 0$ this is obvious in view of $B_{p\rho}^{s} = L_p \cap \dot{B}_{p\rho}$ for $s > 0$). Now it follows from Theorem A that $I^{\alpha}\psi_{jk}^{[l]} \in \dot{B}_{22}^{0}(\mathbb{R}^n) = L_2(\mathbb{R}^n)$. Therefore, $g_\nu \in L_2(\mathbb{R}^n)$, $\nu = 1, 2$. Hence, Plancherel's theorem and the polarization identity

$$\begin{aligned}\langle g_1, g_2\rangle \ &= \frac{1}{4}\Big[\Big(||g_1+g_2||_{L_2(\mathbb{R}^n)}^2 - ||g_1-g_2||_{L_2(\mathbb{R}^n)}^2\Big)\\ &\quad -i\Big(||g_1+ig_2||_{L_2(\mathbb{R}^n)}^2 - ||g_1-ig_2||_{L_2(\mathbb{R}^n)}^2\Big)\Big]\end{aligned}$$

(i being the imaginary unit) imply that

$$\langle g_1, g_2\rangle = \langle Fg_1, Fg_2\rangle \tag{17}$$

holds. Applying (17) to (7) yields

$$c_{jk,j_1æ}^{[l\lambda]} = \langle |.|^{-\alpha} F(\psi_{jk}^{[l]}), F(\psi_{j_1æ}^{[\lambda]})\rangle. \tag{18}$$

Using the fact that $\psi_{jk}^{[l]}$ and $\psi_{j_1æ}^{[\lambda]}$ are tensor products of univariate functions, applying the formulae about the Fourier transform of a dilated and translated univariate function in each of the n coordinates, and making a respective linear change of each of the n integration variables of the n-fold integral defining the duality functional, we obtain from (18) and (17),

$$c_{jk,j_1æ}^{[l\lambda]} = 2^{-j_1\alpha}\langle I^{\alpha}\psi^{[l]}, \psi_{j-j_1,k-2^{j-j_1}æ}\rangle. \tag{19}$$

Next, we consider $h_1 = \psi_{jk}^{[l]}$, $h_2 = I^{\alpha}\psi_{j_1æ}$, and obtain

$$\langle h_1, h_2\rangle = \langle Fh_1, Fh_2\rangle \tag{20}$$

analogously to (17), and, invoking also the fact that $I^{\alpha}\psi^{[\lambda]}$ and $\psi_{j_1-j,æ-2^{j_1-j}k}$ are real-valued functions, we get

$$\begin{aligned}c_{jk,j_1æ}^{[l\lambda]} \ &= 2^{-j\alpha}\langle \psi_{j_1-j,æ-2^{j_1-j}k}^{[l]}, I^{\alpha}\psi^{[\lambda]}\rangle\\ &= 2^{-j\alpha}\langle I^{\alpha}\psi^{[\lambda]}, \psi_{j_1-j,æ-2^{j_1-j}k}^{[l]}\rangle,\end{aligned} \tag{21}$$

analogously to (19). Besides, it is easily seen that

$$\langle Fg_1, Fg_2\rangle = \langle Fh_1, Fh_2\rangle \tag{22}$$

holds. Now (8) follows from (17), (20), (22), (19) and (21). This completes the proof of part (i) of the theorem.

To prove (10) and (12), it suffices to notice that, since $\psi^{[l]} \in B^r_{p\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^r_{p\infty}(\mathbb{R}^n)$ for any: $p:\ 0 < p \le \infty$ and any $l = 1, \dots, 2^n - 1$, Theorem A implies that $I^\alpha\psi^{[l]} \in \dot{B}^{r+\alpha}_{p\infty}(\mathbb{R}^n)$, and, hence, (10) and (12) follow from (6) for $p = \infty$ and for $p:\ 0 < p < \infty$, respectively, with $q = \infty$ in both cases. The proof of part (ii) of the theorem is now complete. ■

Theorem 1 justifies the following definition of generalized FIDO in homogeneous Besov spaces. A generalized FIDO of order $\alpha \in \mathbb{R}$ is any linear operator L^α which is given by an invertible matrix satisfying the conclusions of Theorem 1, with respect of a wavelet basis satisfying the premises of the same theorem. It can be shown that, if $f \in \dot{B}^s_{pq}(\mathbb{R}^n)$, $1 \le p \le \infty$, $0 < q \le \infty$, $|\alpha| < s < r - |\alpha|$, then (i) $L^\alpha L^{-\alpha} f = L^{-\alpha} L^\alpha f$ modulo n-variate polynomials of total degree $< r$; and (ii) L^α has the lifting property. One possible L^α, other than I^α, is the diagonal operator defined by $L^\alpha \phi^{[0]}_{0k} = \phi^{[0]}_{0k}$, $L^\alpha \psi^{[l]}_{jk} = 2^{-\alpha j}\psi^{[l]}_{jk}$.

The diagonal operators L^α_1 and L^α_2 corresponding to two different wavelet bases are unitarily equivalent: there exists a unitary operator $T:\ T^* = T^{-1}$ (T^* being the adjoint of T), such that $L^\alpha_2 = T^* L^\alpha_1 T$.

If both wavelet bases have sufficient regularity (say, if the scaling function ϕ and the wavelet ψ for each of the two bases are in $B^r_{\infty\infty}(\mathbb{R})$, and, given α, r is large enough), then T and T^* are bounded operators on $\dot{B}^s_{pq}(\mathbb{R}^n)$ and $\dot{B}^{s+\alpha}_{pq}(\mathbb{R}^n)$, respectively. (This follows by similar arguments as the ones used in the proof of Theorem 1.) This fact gives additional flexibility in the use of generalized FIDO appropriate for solving (exactly or approximately) a given operator equation.

Theorem 1 is also applied in the proof of the new results about the Riesz potential formulated in Theorem 2 and 3 below.

THEOREM 2. *(Generalization of the lifting property of I^α) Let $\alpha \in \mathbb{R}$ and assume that $1 \le p \le q \le \infty$ and that*

$$\max(0, -r - \alpha + \frac{n}{q}, \frac{n}{q'} - \alpha) < s < \min(r, r - \alpha + \frac{n}{q'}). \tag{23}$$

Then,

$$I^\alpha:\ \dot{B}^s_{pp}(\mathbb{R}^n) \to \dot{B}^{s+\alpha-n(\frac{1}{q'}-\frac{1}{p'})}_{qp}(\mathbb{R}^n),\ 1 < p \le \infty, \tag{24}$$

$$I^\alpha:\ \dot{B}^s_{pp}(\mathbb{R}^n) = \dot{B}^s_{11}(\mathbb{R}^n) \to \dot{B}^{s+\alpha-n(\frac{1}{q'}-\frac{1}{p'})}_{qq}(\mathbb{R}^n) = \dot{B}^{s+\alpha-\frac{n}{q'}}_{qq}(\mathbb{R}^n),\ p = 1. \tag{25}$$

Here $p':\ \frac{1}{p} + \frac{1}{p'} = 1$, $q':\ \frac{1}{q} + \frac{1}{q'} = 1$.

Before giving the proof of this theorem, we note that the standard lifting property of I^α corresponds to the partial case $p = q$.

P r o o f. First we prove the case $p = 1$. Let $\sigma = s + \alpha - \frac{n}{q'}$ and $f \in \dot{B}^s_{11}(\mathbb{R}^n)$. For s and σ the bounds $0 < s < r$ and $0 < \sigma < r$ hold (the bound on σ is true because, by assumption about s, $\frac{n}{q'} - \alpha < s < r - \alpha + \frac{n}{q'}$ is fulfilled). Therefore, $||f||_{\dot{B}^s_{11}(\mathbb{R}^n)}$ and $||I^\alpha f||_{\dot{B}^\sigma_{qq}(\mathbb{R}^n)}$ can be computed via the equivalent seminorms given in (6). We rely on this in the chain of inequalities given below, where we also make use of the following facts:

- the generalized Minkowski inequality;
- formula (12) in Theorem 1, for $p=1$;
- the inequality $\sigma-\alpha-r<0$ is always fulfilled (because $s-\frac{n}{q'}\le s<r$ for any $q:\ 1\le q\le\infty$);
- the inequality $\sigma q+nq-2n+qr>0$ holds true, in view of the assumption $s>-r-\alpha+\frac{n}{q}$ in (23).

$$
\begin{aligned}
&||I^\alpha f||_{\dot{B}^\sigma_{qq}(\mathbb{R}^n)} \asymp \\
&\asymp \left[\sum_{j_1=-\infty}^{\infty}\sum_{æ\in\mathbb{Z}^n}\sum_{\lambda=1}^{2^n-1} 2^{j_1[\sigma+n(\frac{1}{2}-\frac{1}{q})]q}\left|\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1} c^{[l\lambda]}_{jk,j_1æ}\beta^{[l]}_{jk}\right|^q\right]^{\frac{1}{q}} \\
&\le \sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1}|\beta^{[l]}_{jk}|.\left[\sum_{j_1=-\infty}^{\infty}\sum_{æ\in\mathbb{Z}^n}\sum_{\lambda=1}^{2^n-1} 2^{j_1[\sigma+n(\frac{1}{2}-\frac{1}{q})]q}|c^{[l\lambda]}_{jk,j_1æ}|^q\right]^{\frac{1}{q}} \\
&\le C_1\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1}|\beta^{[l]}_{jk}|.\Bigg[2^{-qj[r-\alpha+n(\frac{1}{2}-\frac{1}{q})]}\sum_{j_1=-\infty}^{j-1}2^{j_1(\sigma q+nq-2n+qr)} \\
&\qquad\qquad +2^{qj[r+n(\frac{1}{2}-\frac{1}{q})]}\sum_{j_1=j}^{\infty}2^{j_1q(\sigma-\alpha-r)}\Bigg]^{\frac{1}{q}} \\
&\le C_2\sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1}|\beta^{[l]}_{jk}|.2^{j[\sigma-\alpha+n(\frac{1}{2}-\frac{1}{q})]}\asymp ||f||_{\dot{B}^{\sigma-\alpha+n/q'}_{11}(\mathbb{R}^n)} \\
&= ||f||_{\dot{B}^s_{11}(\mathbb{R}^n)}.
\end{aligned}
$$

This proves (25) and completes the case $p=1$.

Now we turn to the case $1<p\le\infty$. We have just proved that

$$I^\alpha:\ \dot{B}^s_{11}(\mathbb{R}^n)\to\dot{B}^{s+\alpha-n/q'}_{qq}(\mathbb{R}^n). \tag{26}$$

On the other hand, by Theorem A,

$$I^\alpha:\ \dot{B}^s_{qq}(\mathbb{R}^n)\to\dot{B}^{s+\alpha}_{qq}(\mathbb{R}^n). \tag{27}$$

Interpolating between (26) and (27) via the real interpolation method of Lions-Peetre, with $\theta=q'/p'\in(0,1)$, yields, after computations

$$I^\alpha:\dot{B}^s_{pp}(\mathbb{R}^n)\to\dot{B}^{s+\alpha-n(\frac{1}{q'}-\frac{1}{p'})}_{qp}(\mathbb{R}^n),$$

i.e., (24). This completes the proof of the case $1<p\le\infty$. ∎

The next theorem relies on Theorem 1 indirectly, via the case $p=1$ of Theorem 2.

THEOREM 3. *(Generalization of Sobolev's theorem about I^α.) Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $s\in\mathbb{R}$, $\max\left[\frac{2p}{p+1},\frac{2(\frac{n}{\alpha}-1)p}{(\frac{n}{\alpha}-1)p+\frac{n}{\alpha}-p}\right]<\rho\le 2$. Let also $q:\ p\le q<\infty$. Then,*

$$I^\alpha:\ \dot{F}^s_{p\rho}(\mathbb{R}^n)\to\dot{F}^\sigma_{q\lambda}(\mathbb{R}^n),$$

where $\sigma:\ \sigma=s+\alpha+n(\frac{1}{q}-\frac{1}{p})$, $\lambda:\ \frac{1}{\lambda}=\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}+\frac{1}{\rho}+\frac{\alpha}{n}(1-\frac{2}{\rho})$.

Before giving the proof, we note that

$$1 < \max\left[\frac{2p}{p+1}, \frac{2(\frac{n}{\alpha}-1)p}{(\frac{n}{\alpha}-1)p+\frac{n}{\alpha}-p}\right] < 2.$$

Sobolev's Theorem B is a partial case of Theorem 3, namely, when $s=0$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $\rho=2$.

P r o o f. By the limiting case $p=1$ of Theorem 2, (26) holds. On the other hand, by Theorem B,

$$I^\alpha:\ L_{p_1}(\mathbb{R}^n)\to L_{q_1}(\mathbb{R}^n), \tag{28}$$

where $0<\alpha<n$, $1<p_1<\frac{n}{\alpha}$, $\frac{1}{q_1}=\frac{1}{p_1}-\frac{\alpha}{n}$.

In view of $L_{p_1}(\mathbb{R}^n)=\dot{F}^0_{p_1 2}(\mathbb{R}^n)$, $L_{q_1}(\mathbb{R}^n)=\dot{F}^0_{q_1 2}(\mathbb{R}^n)$ in (28) and $\dot{B}^{s_1}_{11}(\mathbb{R}^n)=\dot{F}^{s_1}_{11}(\mathbb{R}^n)$, $\dot{B}^{s_1+\alpha-n/q_2'}_{q_2q_2}(\mathbb{R}^n)=\dot{F}^{s_1+\alpha-n/q_2'}_{q_2q_2}(\mathbb{R}^n)$ in (26) (the equalities are in the sense of isomorphism between the spaces, with equivalent norms), we can rewrite (26) and (28) as

$$I^\alpha:\ \dot{F}^{s_1}_{11}(\mathbb{R}^n)\to \dot{F}^{s_1+\alpha-n/q_2'}_{q_2q_2}(\mathbb{R}^n), \tag{29}$$

$$I^\alpha:\ \dot{F}^0_{p_1 2}(\mathbb{R}^n)\to \dot{F}^0_{q_1 2}(\mathbb{R}^n), \tag{30}$$

respectively. Now we interpolate between (29) and (30) by the complex interpolation method $C_{[\theta]}$ of Calderon-Krein-Lions, $0<\theta<1$, and obtain

$$I^\alpha:\ \dot{F}^s_{p\rho}(\mathbb{R}^n)\to \dot{F}^\sigma_{q\lambda}(\mathbb{R}^n),$$

where

$$\begin{aligned}
&s=(1-\theta)s_1+\theta.0=(1-\theta)s_1,\\
&\tfrac{1}{p}=\tfrac{1-\theta}{1}+\tfrac{\theta}{p_1},\\
&\tfrac{1}{\rho}=\tfrac{1-\theta}{1}+\tfrac{\theta}{2},\\
&\sigma=(1-\theta)(s_1+\alpha-\tfrac{n}{q_2'})+\theta.0=(1-\theta)(s_1+\alpha-\tfrac{n}{q_2'}),\\
&\tfrac{1}{q}=\tfrac{1-\theta}{q_2}+\tfrac{\theta}{q_1},\\
&\tfrac{1}{\lambda}=\tfrac{1-\theta}{q_2}+\tfrac{\theta}{2},\\
&\tfrac{1}{q_1}=\tfrac{1}{p_1}-\tfrac{\alpha}{n}.
\end{aligned}$$

This is a system of 7 equations with 7 unknowns: $\theta, p_1, q_1, q_2, s_1, \sigma, \lambda$; the parameters p, q, s, ρ, α are known. The system has a unique solution which we find below.

From the equality about $\frac{1}{\rho}$, we find $\theta=\frac{2}{\rho'}\in(0,1)$, hence, $s_1=\frac{s\rho}{2-\rho}$, $p_1=\frac{2p/\rho'}{1-(1-2/\rho')p}$. Now the constraint $p_1>0$ implies $1-(1-2/\rho')p>0$, and $\rho>\frac{2p}{p+1}$. Moreover, the constraint $p_1<\frac{n}{\alpha}$ implies, after some computations,

$$\frac{2(\frac{n}{\alpha}-1)p}{(\frac{n}{\alpha}-1)p+\frac{n}{\alpha}-p}<\rho.$$

For q_1 we obtain

$$\frac{1}{q_1}=\frac{1}{p_1}-\frac{\alpha}{n}=\frac{1-(1-2/\rho')p}{2p/\rho'}-\frac{\alpha}{n},$$

and for q_2 we can compute

$$\frac{1-\theta}{q_2} = \frac{1-2/\rho'}{q_2} = \frac{1}{q} - \frac{2/\rho'}{q_1}.$$

Now, after computations, we obtain

$$\sigma = s + \alpha + n(\tfrac{1}{q} - \tfrac{1}{p}),$$
$$\tfrac{1}{\lambda} = \tfrac{\alpha}{n} + \tfrac{1}{q} - \tfrac{1}{p} + \tfrac{1}{\rho} + \tfrac{\alpha}{n}(1 - \tfrac{2}{\rho}).$$

■

Theorems 2 and 3 give conditions for boundedness of I^α when the metric indices p of the domain space and q of the image space are related by $p \le q$. Is it possible in general for I^α to be bounded when in this context $p > q$? The "only if" part of Theorem B shows that the reply to this question is negative. Is this negative result strictly specific for the Riesz potential, or is it characteristic for FIDO (of convolution type) as a whole? The following two results, taken together, show that the condition $p \le q$ is essential for the boundedness of any FIDO (of convolution type).

THEOREM C. *(Stepanov [15].) Let $T: L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$, $1 \le p \le \infty$, $1 \le q \le \infty$, be an integral operator. Then, T is a convolution operator if and only if T commutes with all translations in $\mathbb{R}^n$.*

THEOREM D. *(Hörmander [10].) If $T: L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$, $1 \le p \le \infty$, $1 \le q \le \infty$, if T commutes with all translations in $\mathbb{R}^n$, and if $p > q$, then T is trivial, i.e., $T = 0$.*

4. Concluding remarks

Our approach, developed in this paper for the case of the Riesz potential in homogeneous function spaces, can be extended, mutatis mutandis, to other types of FIDO in the same and other types of function spaces. For example, for $n = 1$ it can be shown that Theorems 1-3 hold, in essentially the same form, for the Riemann-Liouville, Grünwald-Letnikov and Caputo FIDO with terminal at $-\infty$ or $+\infty$. The periodic case can be handled analogously by considering periodized wavelets (see, e.g. Dahmen [5]). Similar (though somehow more technically involved) theory can be developed for the Bessel potential in inhomogeneous Besov and Lizorkin-Triebel spaces over $\mathbb{R}^n$. Atomic decomposition via (bi)orthonormal wavelets has been derived also for Besov spaces on the interval [2,4] and on the hypercube in $\mathbb{R}^n$, and recently Cohen, Dahmen and DeVore [3] have extended these results for arbitrary Lipschitz-graph domains. This opens possibilities to consider generalized FIDO operators with initial and boundary-value conditions on general Lipschitz-graph domains and to derive analogues of Theorems 1-3 for them with applications to the study of regularity of the solutions of initial and boundary-value problems for operator equations on these domains, and for deriving error estimates for their approximate solutions. From this point of view, wavelet methods should be considered as a serious additional tool in the toolkit of methods for numerical solution of fractional operator equations (see [12]).

As far as developing fractional calculus based on the wavelet transform is concerned, the respective results strongly depend on the concrete wavelet chosen. Here we restrict ourselves only to two examples which do not depend on the concrete choice of the underlying wavelet. The first example concerns solving the equations arising in the optimization problem for explicit computation of Peetre's K-functional between Besov spaces, as considered by Dechevsky and Ramsay [7], Appendix B, item B9, (a). The second example is related to fractional differentiation and integration of dyadically self-similar fractals, i.e., functions f which satisfy a functional equation of the type

$$f(x) = \sum_{k=-L}^{L} c_k f(2x-k), \ x \in \mathbb{R}, \ c_k \in \mathbb{R} \text{ or } c_k \in \mathbb{C}, \tag{31}$$

where $L \in \mathbb{N}$ or $L = +\infty$. One important reason why wavelets are very appropriate for studying fractional differentiation and integration of self-similar fractal functions is that both the scaling function ϕ and the wavelet ψ themselves satisfy respective functional equations of type (31).

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