Some Wavelet Analyses of Point Process Data

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Abstract

Point processes may be described by conditional intensity functions and moments. Wavelet analysis provides a means of parameterizing such quantities in the nonstationary case. The parameters of the wavelet expansion may themselves be estimated by the method of moments or likelihood analysis. These ideas are illustrated for data sets arising from nerve cells firing and earthquakes occuring. In particular wavelet parameterized rate, autointensity and conditional intensity functions are estimated. [1]

1. Introduction

Statistical aspects of wavelet analysis for time series and images, both first- and second-order analysis now are fairly well-developed, see [9, 13]. Among the uses of wavelet analysis are: smoothing, estimation, trend analysis, detection of change, detection of jumps, classification, data compression, exploratory analysis, assessing long term dependence and model validation. Basic inferential aspects include regression modelling, perhaps followed by improved estimation via shrinkage. This modification has the property of damping down highly variable terms occurring in an expansion of a function in terms of elementary functions. The idea of shrinkage was employed effectively for crystalography in [1]. There are a variety of ways to introduce wavelet analysis into work with random processes. One is to make parameters appearing time dependent with representations by wavelet expansions. This approach will be employed below.

2. Wavelets for time series

2.1. Introduction

By wavelet analysis will be meant the expansion of functions in the form

$$S(t) \; = \; \sum_j \sum_k \beta_{jk} \psi_{jk}(t) \tag{1}$$

where

$$\psi_{jk}(t) \; = \; 2^{j/2} \psi(2^j t - k) \tag{2}$$

and $0 \; < \; t \; < \; T$. The function $\psi(.)$ is called the mother function. In the analyses of this paper it will be taken to be the Haar function $\psi(t) \; = \; 1 \;\; 0 \leq t < T/2$ and $-1 \;\; T/2 \leq t < T$. In the case of continuous time and with the ψ_{jk} functions orthogonal

$$\beta_{jk} \; = \; \int_0^T \psi_{jk}(t) S(t) dt / \int_0^T \psi_{jk}(t)^2 dt \tag{3}$$

Step (3) expresses analysis and step (1) synthesis.

In practice one may assume the model $Y(t) \; = \; S(t) \; + \; noise$, and then evaluate the estimate $\hat{\beta}_{jk}$ and employ a restricted number of terms in the sample analog of (1). In the empirical case it may be useful to include shrinkage of the coefficients, that is to replace a coefficient $\hat{\beta}_{jk}$ by a shrunken version.

There are also expansions for function of two variables, see [6, 10] and Section 4.2 .

3. Point processes

A univariate temporal point process refers to a sequence of irrregularly distributed times, often corresponding to the moments of occurrence of some particular event of interest.

Supposing that $N(t)$ counts the number of points in the interval $(0, t]$ and that the points of the process are distinct, it is often convenient to represent the process via the differential element

$$dN(t) \;=\; 1\; if\ a\ point\ in\ (0, 0+dt] \quad (4)$$

$$\;=\; 0\ otherwise \quad (5)$$

3.1. Ways to describe point processes

A fundamental description of a point process is provided by the $conditional\ intensity$ function

$$Prob\{dN(t) \;=\; 1|H_t\} \;=\; \mu(t|H_t)dt \quad (6)$$

where $H_t = \{N(u),\ u \leq t\}$. The function $\mu(.)$ provides the intensity with which points are occuring just after time t, given what has already occurred. This function is discussed in [11] for example.

A function describing the first order properties of a point process is provided by the $rate$ or $intensity$ function, $p_N(t)$ of

$$E\{dN(t)\} \;=\; p_N(t)dt \quad (7)$$

The $autointensity$ function is defined by

$$Prob\{dN(t+u) \;=\; 1|dN(t) \;=\; 1\} \;=\; h_{NN}(u)du \quad (8)$$

in the stationary case $u > 0$. Higher-order moment and cumulant functions may be similarly introduced, see [2, 3].

3.2. A particular case

A direct way to introduce serial dependence into a point process model and to be able to use existing computer packages is the following. For some small δ approximate the point process by an integer-valued discrete time series. Then given the history, $H_l = \{Y_m, m \leq l\}$, suppose for example that Y_{l+1} is Poisson with parameter

$$exp\{\gamma + \sum_{m=1}^{M} \alpha_m Y_{l-m}\} \quad (9)$$

where $Y_l = N((l+1)\delta) - N(l\delta)$. Likelihood analysis and programs such as glm() of Splus, [6], may be employed to estimate these α's. Phenonena such as clustering and approximate periodicity may be introduced by choice of the α's and M. For δ small, the process will be essentially $0-1$ valued, [4].

Other link functions than exp might be employed in (9).

4. Wavelets for point processes

There are several ways to introduce wavelet techniques into point process analysis. One is to simply work with the sequence of intervals between events. This paper will not direct attention to this case as time series techniques are immediately available and because of lack of space.

4.1. The rate function

Consider the problem of estimating the function $p_N(t)$ of (7). This is analagous to the problem of estimating a density function, which has been addressed in [8], for example.

One can proceed by expanding the rate function as

$$E\{\frac{dN(t)}{dt}\} \;=\; p_N(t) \;=\; \sum_j \sum_k \beta_{jk}\psi_{jk}(t) \quad (10)$$

and then estimating the β's via

$$\hat{\beta}_{jk} \;=\; \int_0^T \psi_{jk}(t)dN(t) / \int_0^T \psi_{jk}(t)^2 dt \quad (11)$$

following (3). The $\hat{\beta}_{jk}$ may be shrunk, that is replaced by

$$w(\hat{\beta}_{jk}/s_{jk})\hat{\beta}_{jk} \quad (12)$$

where s_{jk} is an estimate of the standard error of $\hat{\beta}_{jk}$ and $w(u)$ is a tapering function. In the examples presented the Tukey function, $(1 - u^{-2})_+$, of [5] is employed.

4.2. The autointensity function

For the stationary case, the autointensity function was defined at (8) and its estimation considered in [2, 3] for example. In the nonstationary case one can consider an expansion for

$$h_{NN}(t,u) \;=\; Prob\{dN(t+u) = 1|dN(t) = 1\}/du \quad (13)$$

based on the functions

$$\psi_{jk}(t)\psi_{j'k'}(u) \quad (14)$$

See [10] for details on wavelet expansions of functions of 2 variables.

4.3. Modeling the conditional intensity

Supposing a point process to have conditional intensity $\mu(t|H_t, \theta)$ including a parameter θ, one can develop a wavelet variant by employing a wavelet expansion for θ.

For example to introduce a wavelet approach into the model (9) one can write the coefficients as

$$\alpha_m(l) \;=\; \sum_j \sum_k \beta_{mjk}\psi_{jk}(l) \quad (15)$$

It can be interesting to see how the α's evolve with time, l.

5. Some examples

5.1. An example from neurophysiology

Figure 1 presents part of the sequence of firing times collected in a neurophysiological experiment. (The experiments in which the data were collected are described in [7].) Figure 2 shows the cumulative count function, $N(t)$, for the whole data set. The figures present an interesting firing behavior; there are periods of silence, then the neuron fires at an increasing rate for a period of time. A preliminary wavelet study is now provided. Figure 3 shows

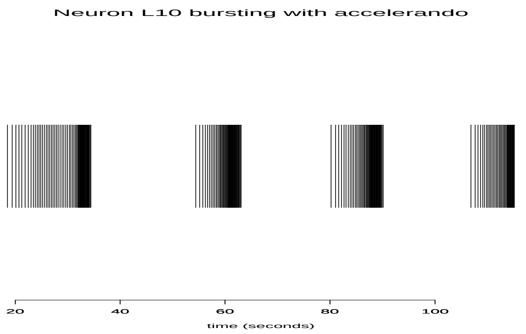

Figure 1. The times an Aplysia neuron fired.

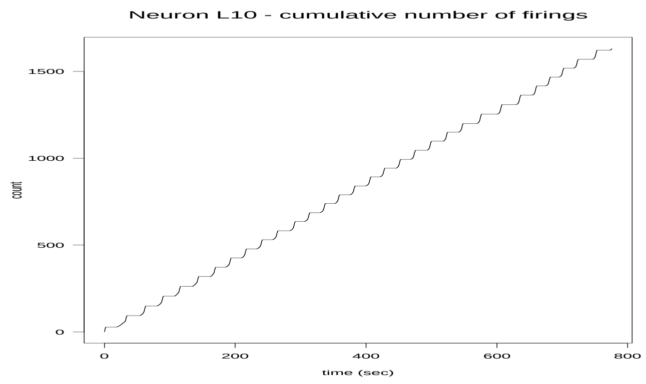

Figure 2. The number of firings at times $\leq t$

wavelet estimates of the time varying rate function of (10). It employs the estimates based on (11) and (12). The lower figure provide approximate marginal ± 2s.e. limits. One sees fluctuations about the overall rate level, with a drop in one interval towards the end.

The autointensity in the stationary case was defined at (7). Figure 4 provides an estimate and also a time varying estimate based on the Haar functions in (14). The top figure highlights the fact that the clusters of firing start about 25 sec, apart with slow variation around this. The lower suggests the interval shrinks towards 20 sec. in the middle. No shrinkage was employed in the estimates here.

5.2. An example from seismology

This section repeats the previous data analyses, but now for a point process taken from seismology.

Figure 5 graphs $N(t)$ versus t for the sequence of occurence times of earthquakes of magnitude 5 or greater as recorded in Northern California for the period 1932-1992. One sees overall stationarity with a crude indication of a higher rate in the middle years followed by a lower rate.

The statistics displayed in Figure 6 are analagous to those of Figure 3 above except that now the autointensity estimate has been further smoothed. On examination one sees a period of higher activity, again followed by one that is lower. Figure 7 shows estimates of the autointensity function in the stationary case and then provides a nonstationary estimate based on Haar wavelets. Again higher levels in the middle are followed by lower later.

In the final computations of this work a model and likelihood analysis of the discrete approximation to the earthquake series is investigated. Specifically the α_m of (9) are replaced by $\alpha_m(l)$ and these are expanded as in (15). Figure 8 graphs the estimated $\alpha_m(l)$ and ± 2 s.e. limits about 0. There is little evidence tha the α's are not all 0.

6. Discussion and summary

Motivated by practical problems, some wavelet based techniques have been suggested for the analysis of point process data. The ability of wavelet analysis to smooth, with a variable binwidth, is a basic characteristic invoked here.

The work of this paper has been concerned with the case of a temporal point process. A variety of extensions are possible to related types of phenomena: eg. spatial point processes, spatial-temporal point processes, marked point processes, hybrids, systems. There are also extensions to wavelet treatments of derived statistics such as the periodogram of spectrum estimation, see [6]. The paper has not provided theory, nor specific asymptotic approximations. Nor have higher-order moments (product densities) been considered, or processes with points of several types, but approaches are possible. It is further remarked that the estimates are preliminary. Parameter choices remain to be investigated, and other data sets to be studied. Standard error limits remain to be computed for some of the estimates.

There is some previous work on point processes using linear expansions in wavelets in searching for long range dependence, see [12].

7. Acknowldegements

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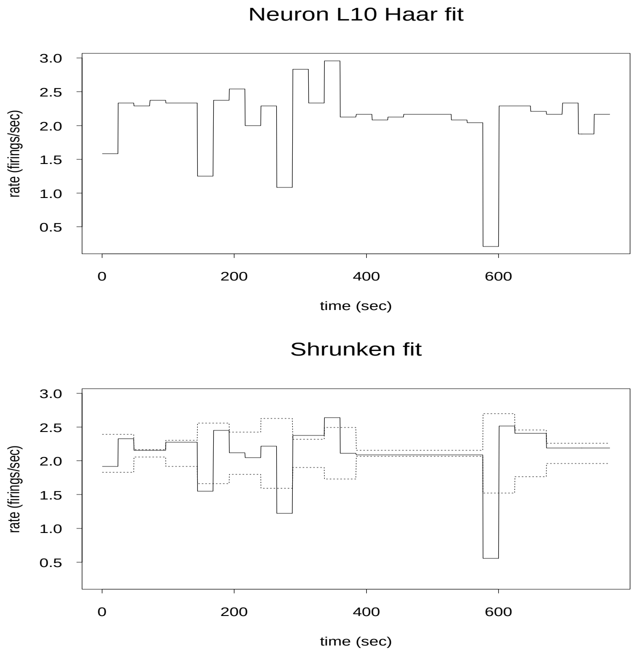

Figure 3. Estimate of time varying rate.

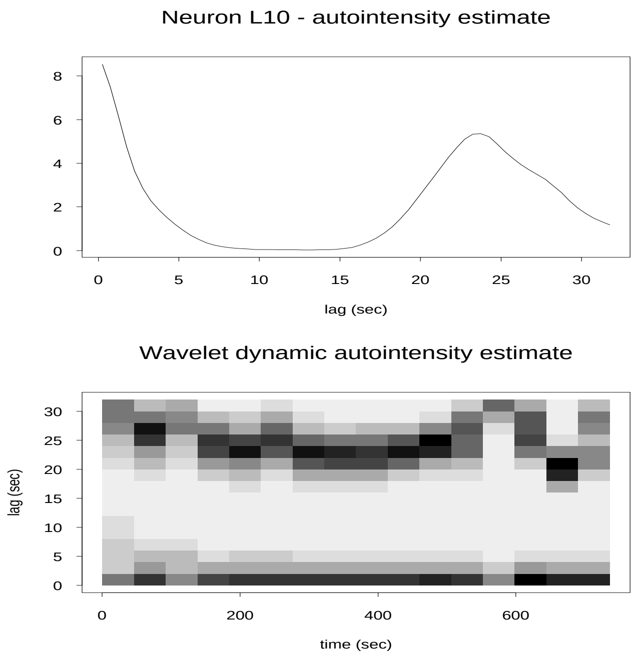

Figure 4. Estimate of time varying autointensity.

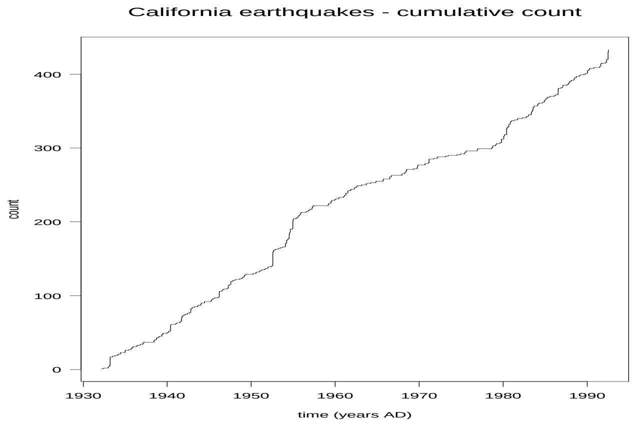

Figure 5. Times of California earthquakes.

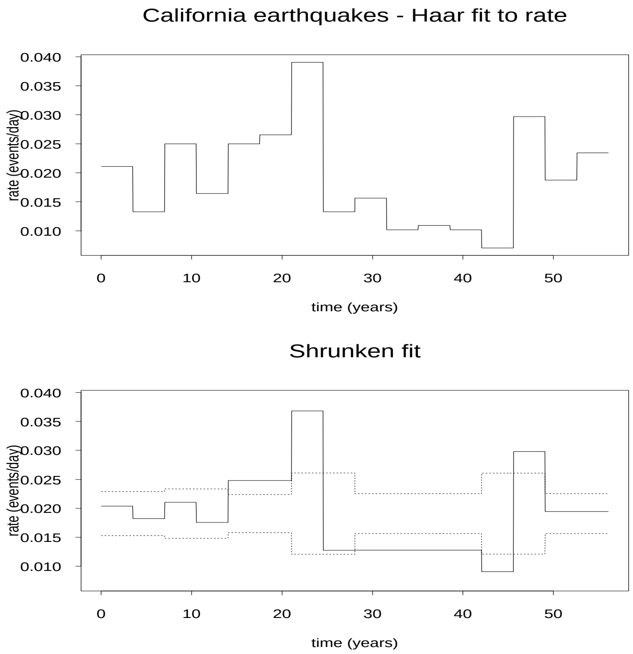

Figure 6. Estimate of time varying rate.

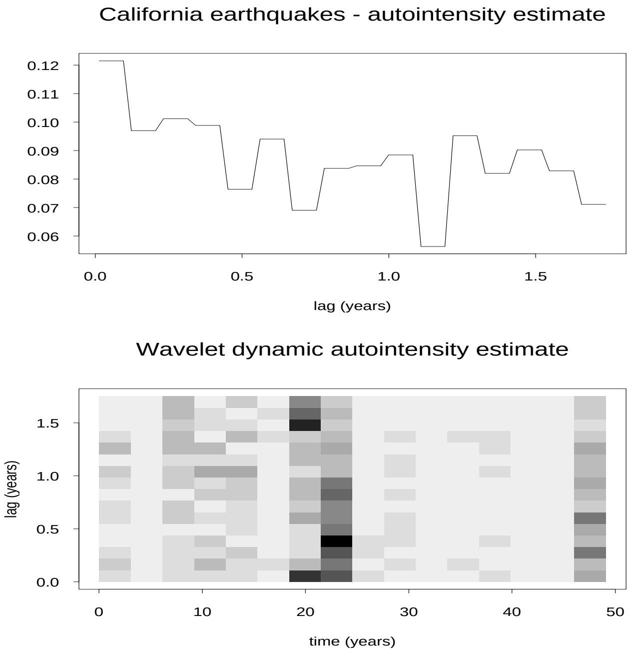

Figure 7. Estimate of time varying autointensity.

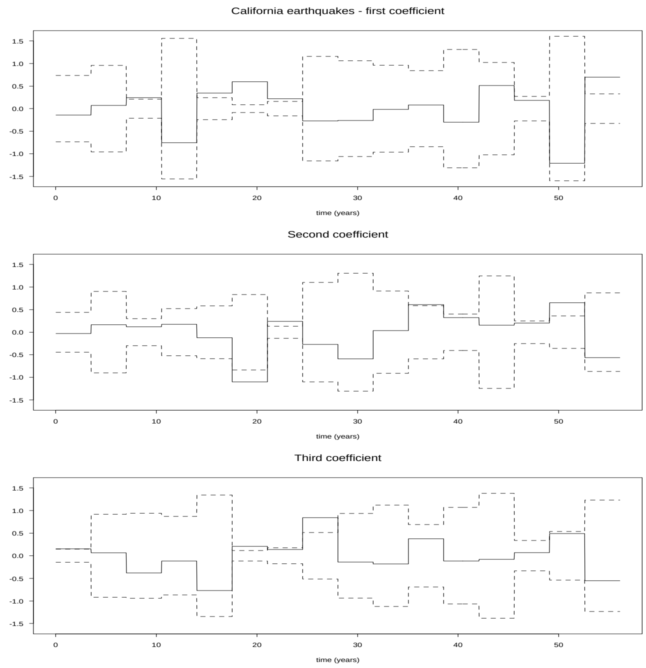

Figure 8. Estimates of the α's of (19).