Solutions to Homework 9, ISyE 2027 Spring 2006

Problem 1

(i) Here

$$Cov(aX, bY) = E[aXbY] - E[aX]E[bY] = abE[XY] - abE[X]E[Y] = abCov(X, Y).$$

(ii) Also

$$Cov(X + Y, U + V) = E[(X + Y)(U + V)] - E[X + Y]E[U + V]$$

$$= E[XU + XV + YU + YV] - E[X]E[U] - E[X]E[V] - E[Y]E[U] - E[Y]E[V]$$

$$= E[XU] - E[X]E[U] + E[XV] - E[X]E[V] + E[YU] - E[Y]E[U] - E[YV] - E[Y]E[V]$$

$$= Cov(X, U) + Cov(X, V) + Cov(Y, U) + Cov(Y, V).$$

Problem 2

(a) $E[3X + 7] = 3E[X] + 7 = 3(3) + 7 = 16.$
$Var(3X + 7) = 9Var(X) = 9(4) = 36.$
(b) $E[2X + 6Y] = 2E[X] + 6E[Y] = 2(3) + 6(-2) = -6$
$Var(2X + 6Y) = 4Var(X) + 36Var(Y) = 4(4) + 36(4) = 160$
(c) $E[5X - 9Z + 8] = 5E[X] - 9E[Z] + 8 = 5(3) - 9(10) + 8 = -67$
$Var(5X - 9Z + 8) = 25Var(X) + 81Var(Z) = 25(4) + 81(2) = 262$
(d) $E[6X + 2Y - Z + 16] = 6(3) + 2(-2) - 10 + 16 = 20$
$Var(6X+2Y-Z+16) = 36Var(X)+4Var(Y)+Var(Z) = 36(4)+4(4)+2 = 162$

Problem 3

(a) We know that the weight of each brick has an expectation of 1.12 kg with a standard deviation of 0.03 kg. If the k^{th} brick selected has a weight of X_k, then the average weight of 25 bricks is just

$$\overline{X}_{25} = \frac{1}{25}\sum_{k=1}^{25} X_k.$$

Hence

$$E[\overline{X}_{25}] = E[\frac{1}{25}\sum_{k=1}^{25} X_k] = \frac{1}{25}\sum_{k=1}^{25} E[X_k] = 1.12,$$

$$Var(\overline{X}_{25}) = \frac{1}{625}\sum_{k=1}^{25} Var(X_k)$$

$$= \frac{1}{625}(25)(.0009) = 0.000036$$

(b) In general,

$$Var(\overline{X}_n) = \frac{0.0009}{n}.$$

We'd like to pick n large enough so that

$$\frac{0.03}{\sqrt{n}} = \sqrt{Var(\overline{X}_n)} \leq 0.005$$

which means that

$$n \geq 36.$$

Problem 4

Throughout this problem, F_X will denote the cdf of X, and f_X the pdf of X.
(a) Suppose $Y = X^3$. Then

$$P(Y \leq t) = P(X^3 \leq t) = P(X \leq t^{1/3}) = F_X(t^{1/3}).$$

After taking derivatives, we see that for $0 \leq t \leq 1$,

$$f_Y(t) = f_X(t^{1/3})(1/3)t^{-2/3} = (2/3)t^{-1/3}.$$

The expected value of Y is just

$$E[Y] = \int_0^1 t(2/3)t^{-1/3}dt = 2/5.$$

(b) Suppose $Y = \sqrt{X}$. Then

$$P(Y \leq t) = P(\sqrt{X} \leq t) = P(X \leq t^2) = F_X(t^2).$$

Again, after taking derivatives, we see that for $0 \leq t \leq 1$,

$$f_Y(t) = f_X(t^2)2t = 4t^3.$$

The expected value of Y is just

$$E[Y] = \int_0^1 4t^4 dt = 4/5.$$

(c) Suppose $Y = 1/(1+X)$. Then

$$P(Y \leq t) = P(1/(1+X) \leq t) = P(X \geq \frac{1}{t} - 1) = 1 - F_X(\frac{1}{t} - 1)$$

so for $1/2 \leq t \leq 1$,

$$f_Y(t) = -f_X(\frac{1}{t} - 1)\frac{-1}{t^2} = \frac{2(1-t)}{t^3}.$$

Then

$$E[Y] = \int_{1/2}^1 \frac{2(1-t)}{t^2} dt = 2 - 2\ln 2.$$

(d) Finally, suppose that $Y = 2^X$.

$$P(Y \leq t) = P(2^X \leq t) = P(X \leq \frac{ln(t)}{ln(2)}) = F_X(ln(t)/ln(2)).$$

Then for $1 \leq t \leq 2$,

$$f_Y(t) = f_X(ln(t)/ln(2))\frac{1}{t\ln 2} = \frac{2\ln t}{t(\ln 2)^2}.$$

The expected value of Y is just

$$E[Y] = \frac{2}{(\ln 2)^2}\int_1^2 \ln(t)dt = \frac{2}{(\ln 2)^2}(2\ln 2 - 2 + 1)$$

$$= 1.608.$$

Problem 5

The mean of Y is just

$$E[Y] = E[3X_1] = 3\mu,$$

and the variance of Y is just

$$Var(Y) = 9Var(X_1) = 9\sigma^2.$$

However, the mean of Z is

$$E[Z] = E[X_1 + X_2 + X_3] = E[X_1] + E[X_2] + E[X_3] = 3\mu$$

but the variance of Z is

$$Var(Z) = Var(X_1) + Var(X_2) + Var(X_3) = 3\sigma^2.$$

We notice that the mean of Y is equal to that of Z, but the variance of Z is much smaller than the variance of Y.

Problem 6

(a) Here

$$1 = \int_0^L Ax(L-x)dx = \frac{AL^3}{6}$$

so it follows that

$$A = \frac{6}{L^3}.$$

(b) The length of one piece is X, and the length of the other piece is $L - X$. Therefore, the difference in the lengths of the two pieces of the rod is just $Y = |X - (L - X)| = |2X - L|$.

The cdf of Y (for $0 \le t \le L$) is

$$P(Y \le t) = P(|2X - L| \le t) = P(-t \le 2X - L \le t)$$

$$= P(2X - L \le t) - P(2X - L \le -t)$$

$$= P(X \le (t + L)/2) - P(X \le (L - t)/2)$$

$$= F_X((L + t)/2) - F_X((L - t)/2).$$

After taking derivatives, we see that the pdf of Y, for $0 \le t \le L$ is

$$f_Y(t) = (1/2)f_X((L + t)/2) + (1/2)f_X((L - t)/2)$$

$$= \frac{3(L^2 - t^2)}{2L^3}.$$

(c) The expected value of Y is

$$E[Y] = \frac{3}{2L^3}\int_0^L t(L^2 - t^2)dt = \frac{3L}{8}$$

Problem 7

Suppose that $Y = aX + b$, where a, b are constants. Notice that

$$Var(Y) = a^2 Var(X)$$

$$Cov(X, Y) = Cov(X, aX + b) = Cov(X, aX) + Cov(X, b) = aVar(X)$$

so the correlation of X and Y is just

$$Corr(X, Y) = \frac{aVar(X)}{\sqrt{Var(X)a^2 Var(X)}} = \frac{aVar(X)}{|a|Var(X)} = \frac{a}{|a|}.$$

If $a > 0$, then $Corr(X, Y) = a/a = 1$. If $a < 0$, then $Corr(X, Y) = a/(-a) = -1$.

Problem 8

(a) Clearly, Z is discrete. The range of Z is just $\{-1, 0, 1, 2, 3, 4, 7, 8\}$. Therefore, the pmf of Z is just

$$P(Z = -1) = 1/8$$

$$P(Z = 0) = 1/4$$

$$P(Z = 1) = 1/8$$

$$P(Z = 2) = 1/8$$

$$P(Z = 3) = 3/16$$

$$P(Z = 4) = 1/16$$

$$P(Z = 7) = 1/16$$

$$P(Z = 8) = 1/16$$

(b) You could do this by saying that

$$E[X - Y^2] = \sum_{i=0}^{3} \sum_{j=0}^{2} (i - j^2) P(X = i, Y = j)$$

and performing the necessary computations. However, notice that X is Binomial with parameters $n = 3$, $p = 1/2$, so $E[X] = 3/2$. Furthermore, Y is Binomial with parameters $n = 2$, $p = 1/2$, so $E[Y^2] = Var(Y) + (E[Y])^2 = 1/2 + 1 = 3/2$. Therefore,

$$E[X - Y^2] = E[X] - E[Y^2] = 3/2 - 3/2 = 0.$$

(c)

$$Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y)$$

$$= 3/4 + 1/2 - 2Cov(X, Y).$$

We showed in class (either on March 15 or March 17) that $Cov(X, Y) = 1/4$, so

$$Var(X - Y) = 3/4 + 1/2 - 1/2 = 3/4.$$