Solutions to Homework 8, ISyE 2027 Spring 2006

Problem 1

Two cards are drawn without replacement from a pack of cards. Let X = the number of heart cards drawn, and let Y = the number of club cards drawn.

(a) We first need to compute the joint pmf of X and Y. Clearly, X and Y can either be 0, 1, or 2. Then

$$P(X=0,Y=0) = \frac{\binom{26}{2}}{\binom{52}{2}} = 0.2451$$

$$P(X=0,Y=1) = \frac{\binom{13}{1}\binom{26}{1}}{\binom{52}{2}} = 0.2549$$

$$P(X=0,Y=2) = \frac{\binom{13}{2}}{\binom{52}{2}} = 0.0588$$

$$P(X=1,Y=0) = \frac{\binom{13}{1}\binom{26}{1}}{\binom{52}{2}} = 0.2549$$

$$P(X=1,Y=1) = \frac{\binom{13}{1}\binom{13}{1}}{\binom{52}{2}} = 0.1275$$

$$P(X=1,Y=2) = 0$$

$$P(X=2,Y=0) = \frac{\binom{13}{2}}{\binom{52}{2}} = 0.0588$$

$$P(X=2,Y=1) = 0$$

$$P(X=2,Y=2) = 0.$$

(b) Now we need to find the marginal probability mass function of X. Here

$$P(X=0) = \sum_{k=0}^{2} P(X=0,Y=k) = 0.5588$$

$$P(X=1) = \sum_{k=0}^{2} P(X=1,Y=k) = 0.3824$$

$$P(X=2) = \sum_{k=0}^{2} P(X=2,Y=k) = 0.0588.$$

Similarly, we can find the marginal pmf of Y:

$$P(Y=0) = \sum_{k=0}^{2} P(X=k,Y=0) = 0.5588$$

$$P(Y=1) = \sum_{k=0}^{2} P(X=k,Y=1) = 0.3824$$

$$P(Y=2) = \sum_{k=0}^{2} P(X=k, Y=2) = 0.0588.$$

(c) X and Y are not independent. To see this, notice that $P(X=2, Y=2) = 0$, but $P(X=2)P(Y=2) = (0.0588)(0.0588) > 0$.
(d)

$$E[X] = \sum_{k=0}^{2} kP(X=k) = 1/2$$

$$E[Y] = \sum_{k=0}^{2} kP(Y=k) = 1/2$$

Also,

$$E[X^2] = \sum_{k=0}^{2} k^2 P(X=k) = 0.6176$$

$$E[Y^2] = \sum_{k=0}^{2} k^2 P(Y=k) = 0.6176$$

so we see that

$$Var(X) = E[X^2] - (E[X])^2 = 0.3676$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = 0.3676.$$

(e) Now

$$E[XY] = \sum_{k=0}^{2}\sum_{l=0}^{2} klP(X=k, Y=l) = 0.1275$$

so now we can compute the covariance:

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 0.1275 - (1/2)(1/2) = -0.1225.$$

(f) Here

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = -0.3332.$$

(g) Let's first compute the conditional pmf of Y, given $X=0$.

$$P(Y=0|X=0) = \frac{P(X=0, Y=0)}{P(X=0)} = \frac{0.2451}{0.5588} = 0.4386$$

$$P(Y=1|X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = 0.4562$$

$$P(Y=2|X=0) = 0.1052.$$

Similarly, the conditional pmf of Y given $X=1$ is

$$P(Y = 0|X = 1) = 0.6666$$

$$P(Y = 1|X = 1) = 0.3334$$

$$P(Y = 2|X = 1) = 0.$$

Problem 2

The joint pdf of X and Y is just

$$f(x, y) = \begin{cases} A(20 - x - 2y), & 0 \le x \le 5,\ 0 \le y \le 5; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Since f is a joint pdf, it must integrate to 1. Therefore,

$$\begin{aligned} 1 = A \int_0^5 \int_0^5 (20 - x - 2y) dy dx &= A \int_0^5 (100 - 5x - 25) dx \\ = A \int_0^5 (75 - 5x) dx &= 5A \int_0^5 (15 - x) dx \\ &= 5A(75 - 25/2) = (625/2)A \end{aligned}$$

so $A = 2/625$.
(b) Here

$$P(1 \le X \le 2, 2 \le Y \le 3) = (2/625) \int_1^2 \int_2^3 (20-x-2y) dy dx = (2/625) \int_1^2 (20-x-5) dx$$

$$= (2/625) \int_1^2 (15 - x) dx = 27/625 = 0.0432.$$

(c) The marginal pdf of X is just

$$f_X(x) = (2/625) \int_0^5 (20 - x - 2y) dy = (10/625)(15 - x)$$

for $0 \le x \le 5$, and 0 otherwise.

Similarly, the marginal pdf of Y is

$$f_Y(y) = (2/625) \int_0^5 (20 - x - 2y) dx = (2/625)(175/2 - 10y)$$

for $0 \le y \le 5$ and 0 otherwise.
(d) X and Y are not independent. Notice that $f_X(0) f_Y(0) = 0.0672$, but $f(0, 0) = 0.064$, so the joint pdf is not equal to the product of the marginal pdfs (which tells us that the two random variables are independent).
(e) Here

$$E[X] = (10/625) \int_0^5 (15x - x^2) dx = 2.333$$

$$E[X^2] = (10/625) \int_0^5 (15x^2 - x^3) dx = 7.5$$

so $Var(X) = E[X^2] - (E[X])^2 = 2.057$.
(f) Similarly,

$$E[Y] = (2/625)\int_0^5 \left(\frac{175}{2}y - 10y^2\right) dy = 2.166$$
$$E[Y^2] = (2/625)\int_0^5 \left(\frac{175}{2}y^2 - 10y^3\right) dy = 6.666$$

so $Var(Y) = E[Y^2] - (E[Y])^2 = 1.974$.
(g) The conditional pdf of Y, given $X = 3$ is

$$f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{(20-3-2y)}{(120/625)} = \frac{625(17-2y)}{120}$$

for $0 \leq y \leq 5$, and 0 otherwise.
(h) Here

$$E[XY] = (2/625)\int_0^5 \int_0^5 xy(20-x-2y)dydx$$

$$= (2/625)\int_0^5 (250x - (25/2)x^2 - (250/3)x)dx$$

$$= 5$$

so the covariance of X and Y is just

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 5 - (2.333)(2.166) = -0.0533.$$

(i)

$$Corr(X,Y) = \frac{-0.0533}{\sqrt{(2.057)(1.974)}} = -0.02644.$$

Problem 3

(a) For $0 \leq x \leq 1$,

$$f_X(x) = \int_0^1 dy = 1$$

and clearly $f_X(x) = 0$ otherwise. Similarly, for $0 \leq y \leq 1$,

$$f_Y(y) = \int_0^1 dx = 1$$

and clearly $f_Y(y) = 0$ otherwise.
(b) The joint pdf is equal to the product of the marginal pdfs, so X and Y are independent.
(c) Notice that $0 \leq X + Y \leq 2$, since $0 \leq X \leq 1$ and $0 \leq Y \leq 1$ with probability 1. Therefore, if $t < 0$, $P(X + Y \leq t) = 0$, and if $t \geq 2$, $P(X + Y \leq t) = 1$.
Suppose that $0 \leq t < 1$. Then

$$P(X + Y \leq t) = \int_0^t \int_0^{-x+t} dydx = t^2/2.$$

But if $1 \le t < 2$, then

$$P(X+Y \le t) = \int_0^{t-1}\int_0^1 dydx + \int_{t-1}^1 \int_0^{t-x} dydx = -(1/2) + t - (t-1)^2/2.$$

(d) Notice that $0 \le XY \le 1$ with probability 1. Therefore, if $t < 0$, $P(XY \le t) = 0$, and if $t \ge 1$, $P(XY \le t) = 1$. If $0 \le t < 1$,

$$P(XY \le t) = \int_0^t \int_0^1 dydx + \int_t^1 \int_0^{t/x} dydx = t - t\ln t.$$

(e) Finally,

$$E[X^2Y] = \int_0^1 \int_0^1 x^2 ydydx = 1/6.$$

Problem 4

(a)

$$P(X \le 2, Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1) + P(X = 2, Y = 1) = 1/5$$

(b)

$$P(X > 2, Y \le 1) = P(X = 3, Y \le 1) = 7/30$$

(c)

$$P(X > Y) = P(X = 1, Y = 0) + P(X = 2, Y = 0) + P(X = 3, Y = 0)$$

$$+P(X = 2, Y = 1) + P(X = 3, Y = 1) + P(X = 3, Y = 2) = 3/5.$$

(d)

$$P(X + Y = 4) = P(X = 3, Y = 1) + P(X = 2, Y = 2) = 8/30 = 4/15.$$

(e) Here

$$P(Y = 0|X = 0) = \frac{P(X = 0, Y = 0)}{P(X = 0)} = 0$$

$$P(Y = 1|X = 0) = \frac{P(X = 0, Y = 1)}{P(X = 0)} = 1/3$$

$$P(Y = 2|X = 0) = 2/3.$$

(f) Also

$$P(X = 0|Y = 2) = 1/7$$

$$P(X = 1|Y = 2) = 3/14$$

$$P(X = 2|Y = 2) = 4/14$$

$$P(X = 3|Y = 2) = 5/14$$

(g) Finally,

$$E[XY] = \sum_{k=0}^{3} \sum_{l=0}^{2} klP(X = k, Y = l) = 2.4.$$

Problem 5

Let X denote the arrival time of person 1, and let Y denote the arrival time of person 2. To allow for clean calculations, let's suppose that X and Y are independent uniform random variables with parameters 0 and 1 (so 0 corresponds to 5 pm and 1 corresponds to 6 pm). Now, for both people to meet, the first person must arrive 15 minutes before or after the second person, so we need to compute $P(|X-Y| \leq 1/4)$ (clearly 1/4 corresponds to 15 minutes).

Now

$$P(|X - Y| \leq 1/4) = \int_0^{1/4} \int_0^{x+1/4} dydx$$
$$+ \int_{1/4}^{3/4} \int_{x-1/4}^{x-3/4} dydx + \int_{3/4}^{1} \int_{x-1/4}^{1} dydx$$
$$= 1 - (1/2)(3/4)^2 - (1/2)(3/4)^2 = 7/16.$$