Solutions to Homework 7, ISyE 2027 Spring 2006

Problem 1

(a) The expected distance between two imperfections is just $1/\lambda = 1/2$ meters.
(b) Here $P(X > 1) = e^{-2} = 0.1353$.
(c) The distribution of the number of imperfections in a 3-meter stretch of fiber is Poisson with rate $2(3) = 6$.
(d)

$$P(N(3) \leq 4) = \sum_{k=0}^{4} \frac{6^k e^{-6}}{k!} = 0.2851.$$

Problem 2

(a) This is a gamma distribution, with parameters $\alpha = 4$ and $\lambda = 2$.
(b) Since $X_2 + X_3 + X_4 + X_5$ is a gamma r.v., we see that

$$E[X_2 + X_3 + X_4 + X_5] = 4/2 = 2.$$

(c) Again, since $X_2 + X_3 + X_4 + X_5$ is a gamma r.v., we find that the variance is just

$$Var(X_2 + X_3 + X_4 + X_5) = 4/2^2 = 1$$

so it follows that the standard deviation of this random variable is 1.
(d) Let $T = X_2 + X_3 + X_4 + X_5$. Once could use integration by parts three times to conclude that

$$P(T > 3) = \int_3^\infty \frac{2(2t)^3 e^{-2t}}{3!} dt = 0.1512.$$

However, notice that T has the same distribution as $S_4 = X_1 + X_2 + X_3 + X_4$ (in other words, their cumulative distribution functions are equal), so

$$P(T > 3) = P(S_4 > 3) = P(N(3) \leq 3)$$

where N is a Poisson process with rate $\lambda = 2$. Thus,

$$P(N(3) \leq 3) = \sum_{k=0}^{3} \frac{6^k e^{-6}}{k!} = 0.1512.$$

Problem 3

Let X_i = the time to complete the i^{th} task, where $i \in \{1, 2\}$. Here X_1 and X_2 are independent.

(a) We're interested in the total time to complete both tasks, which is just $X_1 + X_2$. We know that this is a gamma random variable with parameters $\alpha = 2$ and $\lambda = 1/10$, so

$$E[X_1 + X_2] = \frac{2}{1/10} = 20$$

and

$$Var(X_1 + X_2) = \frac{2}{(1/10)^2} = 200.$$

(b) The average time to compute the two tasks is just $(X_1 + X_2)/2$, so

$$E[(X_1 + X_2)/2] = (1/2)E[X_1 + X_2] = 10$$

and

$$Var((X_1 + X_2)/2) = (1/4)Var(X_1 + X_2) = 50.$$

Problem 4

Because cars arrive in accordance with a Poisson process, we can assume that Shep begins crossing the road at time zero. If we suppose that it takes the dog s seconds to cross the road, then the probability that he is uninjured is just

$$P(N(s/60) = 0) = e^{-\lambda s/60} = e^{-s/20}$$

since the arrival rate is just $\lambda = 3$. Now

$$P(N(5/60) = 0) = e^{-1/4} = 0.7788$$

$$P(N(10/60) = 0) = e^{-1/2} = 0.6065$$

$$P(N(20/60) = 0) = e^{-1} = 0.3679$$

$$P(N(30/60) = 0) = e^{-3/2} = 0.2231.$$

Problem 5

(a) $E[X_1] = 1/5.2 = 0.1923$.
(b) $P(X_1 > 1/6) = e^{-(5.2)(1/6)} = 0.4204$.
(c) This is a gamma distribution with parameters $\alpha = 10$ and $\lambda = 5.2$.
(d) Let $S_10 = X_1 + X_2 + ... + X_10$. Then S_10 is gamma (from part (c)), so

$$E[S_10] = 10/5.2 = 1.923.$$

(e)

$$P(N(1) > 5) = 1 - P(N(1) \leq 5) = 1 - \sum_{k=0}^{5} \frac{(5.2)^k e^{-5.2}}{k!}$$

$$= 0.4191.$$

Problem 6

Here we are given that X is a $N(-7, 14)$ r.v.. Let Z be a standard normal random variable (i.e. a $N(0, 1)$ r.v.). Recall from your lecture notes that X is equal in distribution to $\sigma Z + \mu$, which means that for each real number t,

$$P(X \leq t) = P(\sigma Z + \mu \leq t).$$

(a) $P(X \leq 0) = P(Z \leq 1.87) = 0.9693$.
(b)

$$P(X \geq -10) = P(Z \geq \frac{-10 + 7}{\sqrt{14}})$$

$$= P(Z \geq -0.802) = P(Z \leq 0.802) = 0.7939.$$

(c)

$$P(-15 \leq X \leq 1) = P(\frac{-15 + 7}{\sqrt{14}} \leq Z \leq \frac{-1 + 7}{\sqrt{14}})$$

$$= P(Z \leq 1.60) - P(Z \leq -2.14) = 0.9452 - 0.0162 = 0.929.$$

(d)

$$P(-5 \leq X \leq 2) = P(0.53 \leq Z \leq 2.41) = 0.9920 - 0.7019 = 0.2901.$$

(e)

$$P(|X + 7| \geq 8) = 1 - P(|X + 7| \leq 8) = 1 - P(-8 \leq X + 7 \leq 8)$$

$$= 1 - P(-\frac{8}{\sqrt{14}} \leq Z \leq \frac{8}{\sqrt{14}})$$

$$= 1 - 0.9838 + 0.0162 = 0.0324.$$

(f) If

$$P(X \leq x) = P(Z \leq \frac{x+7}{\sqrt{14}}) = 0.75,$$

we can look in the table to see that $\frac{x+7}{\sqrt{14}} = 0.675$, or $x = -4.47$.

(g) If $P(X \geq x) = 0.27$, then

$$P(X \leq x) = P(Z \leq \frac{x+7}{\sqrt{14}}) = 0.73,$$

which means that $\frac{x+7}{\sqrt{14}} = 0.61$, or $x = -4.72$.

(h) Notice that

$$0.63 = P(|X+7| \geq x) = 1 - P(|X+7| \leq x) = 1 - P(-\frac{x}{\sqrt{14}} \leq Z \leq \frac{x}{\sqrt{14}})$$

$$= 1 - P(Z \leq \frac{x}{\sqrt{14}}) + P(Z \leq -\frac{x}{\sqrt{14}})$$

$$= P(Z \geq \frac{x}{\sqrt{14}}) + P(Z \leq -\frac{x}{\sqrt{14}})$$

$$= P(Z \leq -\frac{x}{\sqrt{14}}) + P(Z \leq -\frac{x}{\sqrt{14}})$$

$$= 2P(Z \leq -\frac{x}{\sqrt{14}})$$

so we see that

$$0.315 = P(Z \leq -\frac{x}{\sqrt{14}})$$

which means that

$$-0.48 = -x/\sqrt{14}$$

or

$$x = 1.80.$$

Problem 7

Since $P(X \leq 10) = 0.55$, we see that $P(\sigma Z + \mu \leq 10) = 0.55$, or $P(Z \leq (10-\mu)/\sigma) = 0.55$. Furthermore, $P(Z \leq -\mu/\sigma) = 0.40$. This means that

$$\frac{10-\mu}{\sigma} = 0.125$$

$$-\frac{\mu}{\sigma} = -0.253.$$

After solving for μ and λ, we see that $\mu = 6.69$, and $\sigma = 26.45$.

Problem 8

Here $P(X \leq 10) = 0.01$, or

$$P(Z \leq \frac{10-\mu}{0.05}) = 0.01$$

This implies that

$$-2.3 = \frac{10-\mu}{0.05}$$

so $\mu = 10.115$.