Solutions to Homework 6, ISyE 2027 Spring 2006

Problem 1

(a) Binomial (b) Geometric (c) Poisson (d) Negative Binomial (e) Bernoulli

Problem 2

Let X = number of people that need to take sick leave. Clearly X is a Binomial random variable with parameters $n = 180$ and $p = 0.35$.
The proportion of those that need to take sick leave is just

$$Y = \frac{X}{180}$$

so we see that

$$E[Y] = \frac{1}{180}(180)(.35) = .35$$

$$Var(Y) = \frac{1}{(180)^2}Var(X) = \frac{(.35)(.65)}{180} = 0.00126.$$

For arbitrary p, the variance of Y is just $p(1-p)/180$. Set $f(p) = p(1-p)/180$. Here $f^{'}(p) = 1/180 - p/90$, so f only has one critical point, namely $p^* = 1/2$. Also, $f^{''}(p^*) = -1/90 < 0$, which tells us that f attains its maximum at p^*. Therefore, the variance is at its largest when $p = 1/2$.

Problem 3

Suppose X is a negative binomial random variable with parameters $r = 3$, and $p = 0.6$.
(a) $P(X = 5) = 0.20736$.
(b) $P(X = 8) = 0.04645$.
(c) $P(X \leq 7) = \sum_{k=3}^{7} P(X = k) = 0.90374$.
(d) $P(X \geq 7) = 1 - P(X \leq 6) = 0.1792$.

Problem 4

(a) Let N_A = the number of fish caught before the first adult fish is caught (including the adult fish). Then N_A is a geometric random variable with parameter $p = 0.77$, so

$$E[N_A] = 1/.77 = 1.2987.$$

(b) Let N_Y = the number of fish caught before the first young fish is caught. Then N_Y is a geometric random variable with parameter $p = 0.23$, so

$$P(N_Y = 5) = (.77)^4(.23) = 0.08085.$$

(c) Let X = the number of fish caught when the fisherman stops for lunch. In this case, we see that X is a negative binomial random variable with parameters $r = 3$ and $p = .77$, so

$$P(X = 6) = 0.05555.$$

(d) Let $Y =$ the number of adult fish among the first eight fish that are caught. Notice that Y is a Binomial random variable with parameters $n = 8$ and $p = .77$. Then

$$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.99732.$$

Problem 5

Let $X =$ the number of white balls found in a sample of size 4 selected at random from the urn. In this case, X is a hypergeometric random variable, and

$$P(X = 2) = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = 0.514.$$

Let $T =$ the number of trials required until we sample two white balls. Clearly T is a geometric random variable with parameter $p = 0.514$, so for any $n \geq 1$,

$$P(T = n) = (.514)(.486)^{n-1}.$$

Problem 6

Let $X =$ the number of passengers that do not show up. Clearly X is a binomial random variable with parameters $n = 200$ and $p = 0.01$. Since n is large and p is small, it is reasonable for us to use our Poisson approximation result. Thus,

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) \approx 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2} = 0.59399.$$

Problem 7

Here X is a Poisson random variable with parameter $\lambda = 4$.
(a) $P(X = 0) = e^{-4} = 0.01832$.
(b) $P(X \geq 6) = 1 - P(X \leq 5) = 0.21487$.

Problem 8

Now X will denote a uniform random variable, with parameters $a = 1.43$ and $b = 1.6$.
(a) $E[X] = (1.43 + 1.6)/2 = 1.515$.
(b) Here $Var(X) = (1/12)(1.6 - 1.43)^2 = 0.00241$, so we see that the standard deviation (the positive square root of the variance) of X is just 0.04909.
(c) The cdf of X is as follows:

$$F(t) = \begin{cases} 0, & t < 1.43; \\ \frac{t-1.43}{1.6-1.43}, & 1.43 \leq t < 1.6; \\ 1, & t \geq 1.6. \end{cases}$$

(d) $P(X \leq 1.48) = (1.48 - 1.43)/(1.6 - 1.43) = 0.29412$.
(e) Notice that $P(X \leq 1.5) = 0.41176$. Let $N =$ the number of batteries in the box with a voltage less than 1.5. Notice that N is a binomial random variable with parameters $n = 50$ and $p = 0.41176$. Then

$$E[N] = (50)(.41176) = 20.59$$

$$Var(N) = (50)(.41176)(.58824) = 12.11.$$