Solutions to Homework 5, ISyE 2027 Spring 2006

Problem 1

Our sample space is just $\Omega = \{(x_1, x_2, x_3) : x_1 \in \{H, T\}, x_2 \in \{1, 2, 3, 4, 5, 6\}, x_3 \in \{1, 2, 3, 4, 5, 6\}\}$, where we interpret (x_1, x_2, x_3) to mean the coin flip resulted in x_1, the red die scored a x_2, and the blue die scored an x_3. Furthermore, the probability of each outcome in our space is just $1/72$, because there are 72 elements in Ω, and each outcome is equally likely (based on our assumptions) to occur.
Let X denote our winnings. Notice that, for example

$$P(X = 1) = P(\{(T, 1, 1), (T, 1, 2), (T, 1, 3), (T, 1, 4), (T, 1, 5), (T, 1, 6)\}) = 6/72,$$

$$P(X = 2) = P(\{(H, 1, 1), (T, 2, 1), (T, 2, 2), (T, 2, 3), (T, 2, 4), (T, 2, 5), (T, 2, 6)\}) = 7/72,$$

$$P(X = 3) = P(\{(H, 1, 2), (H, 2, 1), (T, 3, 1), (T, 3, 2), (T, 3, 3), (T, 3, 4), (T, 3, 5), (T, 3, 6)\}) = 8/72.$$

If one applies this argument to all other cases, we see that

$$P(X = 4) = 9/72$$

$$P(X = 5) = 10/72$$

$$P(X = 6) = 11/72$$

$$P(X = 7) = 6/72$$

$$P(X = 8) = 5/72$$

$$P(X = 9) = 4/72$$

$$P(X = 10) = 3/72$$

$$P(X = 11) = 2/72$$

$$P(X = 12) = 1/72.$$

Therefore, we find that

$$E[X] = \sum_{k=1}^{12} kP(X = k) = 5.25.$$

Problem 2

(a)

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_4^6 \frac{1}{ln(1.5)}dx = \frac{2}{ln(1.5)} = 4.933.$$

(b) We know that X is a continuous random variable, so we need to find all the values m such that $F(m) = 1/2$. Notice that

$$F(t) = \begin{cases} 0, & t < 4; \\ \frac{ln(t)-ln(4)}{ln(6)-ln(4)}, & 4 \leq t < 6; \\ 1, & t \geq 6. \end{cases}$$

so it suffices to find all values m that satisfy

$$\frac{ln(m) - ln(4)}{ln(6) - ln(4)} = 1/2,$$

or

$$ln(m) = ln(4) + (ln(3/2))/2,$$

or

$$m = e^{ln(4)+(ln(3/2))/2}.$$

This equation has only one solution, and it is $m = 4.899$.

Problem 3

(a) Recall from your class notes that for any function g,

$$E[g(X)] = \sum_k g(k)P(X = k).$$

In this case, $g(k) = min(k, 3)$, so

$$E[min(X, 3)] = \sum_{k=1}^{6} min(k, 3)P(X = k) = P(X = 1) + 2P(X = 2) + 3P(X \geq 3)$$

$$= 1/21 + 4/21 + 3(18)/21 = 59/21 = 2.810.$$

(b) Now our cumulative distribution function is just

$$F(t) = \begin{cases} 0, & t < 1; \\ 1/21, & 1 \leq t < 2; \\ 3/21, & 2 \leq t < 3; \\ 6/21, & 3 \leq t < 4; \\ 10/21, & 4 \leq t < 5; \\ 15/21, & 5 \leq t < 6; \\ 1, & t \geq 6. \end{cases}$$

so $\{t : P(X \leq t) \geq 1/2\} = [5, \infty)$. Also, we see that $\{t : P(X \geq t) \geq 1/2\} = (-\infty, 5]$, since

$$P(X \geq 5) = P(X = 5) + P(X = 6) = 11/21 > 1/2,$$

and for $t > 5$

$$P(X \geq t) = P(X = 6) = 6/21 < 1/2.$$

Now, the set of all medians is just the set of values m that satisfy $P(X \leq m) \geq 1/2$, and $P(X \geq m) \geq 1/2$, which is just $\{t : P(X \leq t) \geq 1/2\} \cap \{t : P(X \geq t) \geq 1/2\}$. Therefore, the set of medians is just $(-\infty, 5] \cap [5, \infty) = \{5\}$, so there is only one median.

(c) Notice that $Var(3X - 4) = 9Var(X)$ (we showed in class that $Var(aX + b) = a^2 Var(X)$). Here

$$E[X] = \sum_{k=1}^{6} kP(X = k) = \sum_{k=1}^{6} k(k/21) = 4.333$$

and

$$E[X^2] = \sum_{k=1}^{6} k^2 P(X = k) = (1/21) \sum_{k=1}^{6} k^3 = 21$$

so we see that $Var(X) = 21 - (4.333)^2 = 2.222$, which means that $Var(3X - 4) = 9(2.222) = 20$.

(d) Recall that the 40th quantile of X (we'll denote this as $x_{.4}$) is just the smallest value of t such that $F(t) \geq .4$. Now $F(4) = 10/21 > .4$, but for any $t < 4$, $F(t) \leq 6/21 < .4$, so we see that $x_{.4} = 4$.

Problem 4

Let X equal the number of students taking ISyE 2027. We know that $E[X] = 150 = \mu$, and $Var(X) = (25)^2 = \sigma^2$. Using Chebyshev's inequality, we see that

$$P(100 \leq X \leq 200) = P(-50 \leq X - 150 \leq 50)$$

$$= P(|X - \mu| \leq 2\sigma) \geq 1 - \frac{1}{2^2} = .75$$

so a positive lower bound for this probability is .75.

Problem 5

(a) Notice that the probability density function is just

$$f(t) = \begin{cases} t/8, & 0 < t < 4; \\ 0, & \text{otherwise.} \end{cases}$$

Now we can calculate the variance of X. Notice that

$$E[X] = \int_0^4 x(x/8)dx = (1/8) \int_0^4 x^2 dx = 8/3$$

and

$$E[X^2] = (1/8) \int_0^4 x^3 dx = 8$$

so

$$Var(X) = 8 - (8/3)^2 = 8/9.$$

(b) The standard deviation is just the positive square root of the variance, which in this case is just 0.9428.
(c) We want to find $x_{.75}$, which is the smallest t satisfying $F(t) \geq .75$. Since F is strictly increasing between 0 and 4, we see that this is equivalent to finding t satisfying $F(t) = .75$.
This means that

$$\frac{(x_{.75})^2}{16} = \frac{3}{4}$$

so it follows that $x_{.75} = \sqrt{12} = 3.464$.
(d) Similarly, we see that $x_{.25} = \sqrt{4} = 2$.

Problem 6

(a) The cdf of X is just

$$F(t) = \begin{cases} 0, & t < 0; \\ 1 - p, & 0 \leq t < 1; \\ 1, & t \geq 1. \end{cases}$$

(b) Suppose $p = 1/2$. Notice that $\{t : P(X \leq t) \geq 1/2\} = [0, \infty)$, and $\{t : P(X \geq t) \geq 1/2\} = (-\infty, 1]\}$. Therefore, the set of all medians is just $(-\infty, 1] \cap [0, \infty) = [0, 1]$ (so here's an example where we have an infinite number of medians).
(c) Now suppose $p < 1/2$. Notice that $\{t : P(X \leq t) \geq 1/2\} = [0, \infty)$, and $\{t : P(X \geq t) \geq 1/2\} = (-\infty, 0]$. In this case, the set of all medians is just $(-\infty, 0] \cap [0, \infty) = \{0\}$, which means that the median is unique. If $0 < p < 1/2$, then clearly the mean and the median are not equal. However, if $p = 0$, then they are equal. When I wrote the problem, I forgot to state that $p > 0$, so I won't penalize you if you didn't state whether or not the mean and the median are equal.

Problem 7

(a) Let X = the number of questions that are answered correctly. Clearly X is a Binomial random variable with parameters $n = 10$ and $p = .2$. Thus,

$$P(X \geq 7) = \sum_{k=7}^{10} \binom{10}{k} (.2)^k (.8)^{10-k} = 0.00086.$$

(b) X is still a Binomial random variable, but now the parameters are $n = 10$ and $p = .5$. In this case,

$$P(X \geq 7) = \sum_{k=7}^{10} \binom{10}{k} (.5)^k (.5)^{10-k} = 0.1719.$$