Solutions to Homework 4, ISyE 2027 Spring 2006

Problem 1

(a) The sample space of this experiment is

$$\Omega = \{(H,H,H),(H,H,M),(H,M,H),(M,H,H),$$

$$(M,M,H),(M,H,M),(H,M,M),(M,M,M)\}$$

where for example (H,M,H) is meant to represent the outcome corresponding to making the first shot, missing the second and making the third.
(b) $\{X=0\} = \{(M,M,M)\}$, and

$$P(X=0) = P(\{(M,M,M)\}) = (.2)(.3)(.4) = 0.024.$$

(c) $\{X=2\} = \{(H,H,M),(H,M,H),(M,H,H)\}$, and

$$P(X=2) = P(\{(H,H,M),(H,M,H),(M,H,H)\})$$

$$= (.8)(.7)(.4) + (.8)(.3)(.6) + (.2)(.7)(.6) = 0.452.$$

(d) Notice that $P(X=3) = (.8)(.7)(.6) = 0.336$. Using this and our answers from parts (b) and (c), we see that

$$P(X=1) = 1 - P(X=0) - P(X=1) - P(X=2) = 0.188.$$

This implies that our cdf is just F, where

$$F(t) = \begin{cases} 0, & t<0; \\ .024, & 0 \le t < 1; \\ .212, & 1 \le t < 2; \\ .664, & 2 \le t < 3; \\ 1, & t \ge 3. \end{cases}$$

How did we compute this? Notice that if $t<0$, then

$$F(t) = P(X \le t) = 0$$

since X can only be either 0, 1, 2, or 3. If $0 \le t < 1$, then

$$F(t) = P(X \le t) = P(X=0) = 0.024$$

since the only way that $X \le t$ is if $X = 0$ (again, X is either 0, 1, 2, or 3). Similarly, if $1 \le t < 2$,

$$F(t) = P(X \le t) = P(X=0) + P(X=1) = 0.212.$$

since the only way that $X \le t$ is if $X = 0$ or $X = 1$. Again, if $2 \le t < 3$,

$$F(t) = P(X \le t) = P(X = 0) + P(X = 1) + P(X = 2) = 0.664.$$

Finally, if $t \ge 3$, then $F(t) = P(X \le t) = 1$ since it is always true that $X \le 3 \le t$.

Problem 2

Let X denote the net winnings. Notice that the range of X is just $\{-1, 0, 1, 3, 4, 5\}$. To see this, notice that if we pick up $\{1, 2\}$, then $X = 1+2-4 = -1$. If we pick up $\{1, 3\}$, then $X = 1 + 3 - 4 = 0$, and similarly for the other four possibilities.

Here $P(X = -1) = P(\{1, 2\}) = 1/6$, $P(X = 0) = P(\{1, 3\}) = 1/6$, $P(X = 1) = P(\{2, 3\}) = 1/6$, $P(X = 3) = P(\{1, 6\}) = 1/6$, $P(X = 4) = P(\{2, 6\}) = 1/6$, and $P(X = 5) = P(\{3, 6\}) = 1/6$. Therefore, our probability mass function (pmf) is just

$$p(-1) = P(X = -1) = 1/6$$

$$p(0) = P(X = 0) = 1/6$$

$$p(1) = P(X = 1) = 1/6$$

$$p(3) = P(X = 3) = 1/6$$

$$p(4) = P(X = 4) = 1/6$$

$$p(5) = P(X = 5) = 1/6.$$

Our cumulative distribution function (cdf) F is just

$$F(t) = \begin{cases} 0, & t < -1; \\ 1/6, & -1 \le t < 0; \\ 1/3, & 0 \le t < 1; \\ 1/2, & 1 \le t < 3; \\ 2/3, & 3 \le t < 4; \\ 5/6, & 4 \le t < 5; \\ 1, & t \ge 5. \end{cases}$$

Problem 3

(a) Since f is a probability density function (pdf), we see that

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 Ax(1-x)dx$$

$$= A(1/2 - 1/3) = A/6$$

so $A = 6$.
(b) If $t < 0$, then we see that $F(t) = \int_{-\infty}^{t} f(x)dx = 0$ since $f(x) = 0$ when $x < 0$. If $0 \leq t < 1$,

$$F(t) = \int_{-\infty}^{t} f(x)dx = \int_{0}^{t} 6x(1-x)dx$$

$$= 3t^2 - 2t^3 = t^2(3-2t).$$

If $t \geq 1$, then $F(t) = \int_{-\infty}^{t} f(x)dx = \int_{0}^{1} f(x)dx = 1$. So to summarize,

$$F(t) = \begin{cases} 0, & t < 0; \\ t^2(3-2t), & 0 \leq t < 1; \\ 1, & t \geq 1. \end{cases}$$

Problem 4

(a) I'll let you sketch it yourself.
(b) $P(X \leq 2) = F(2) = \frac{2^2}{16} = 1/4$.
(c) $P(1 \leq X \leq 3 = P(X \leq 3) - P(X < 1) = P(X \leq 3 - P(X \leq 1) = F(3) - F(1) = 9/16 - 1/16 = 1/2$. The second equality is true because F is continuous at 1.
(d) The density function is just the derivative of the distribution function. If $t < 0$ or $t > 4$, then we see that the derivative of F is just 0. If $0 < t < 4$, then $f(t) = 2x/16 = x/8$, so our density is just

$$f(t) = \begin{cases} x/8, & 0 < t < 4; \\ 0, & \text{otherwise.} \end{cases}$$

Again, I'll let you sketch the density function yourself.

Problem 5

(a) $P(X > 1/2) = \int_{1/2}^{\infty} f(x)dx = \int_{1/2}^{1} 2xdx = 1 - 1/4 = 3/4$.
(b)

$$P(X > 1/2 | X > 1/4) = \frac{P(X > 1/2, X > 1/4)}{P(X > 1/4)}$$

$$= \frac{P(X > 1/2)}{P(X > 1/4)} = \frac{12/16}{15/16} = 4/5.$$

(c)

$$P(X > 1/4 | X > 1/2) = \frac{P(X > 1/4, X > 1/2)}{P(X > 1/2)}$$

$$= \frac{P(X > 1/2)}{P(X > 1/2)} = 1.$$

(d) If $t < 0$, then $F(t) = \int_{-\infty}^{t} f(x)dx = 0$ since $f(x) = 0$ if $x < 0$. If $0 \leq t < 1$,

$$F(t) = \int_{-\infty}^{t} f(x)dx = \int_{0}^{t} 2xdx = t^2.$$

Finally, if $t \geq 1$, then $F(t) = \int_{-\infty}^{t} f(x)dx = \int_{0}^{1} 2xdx = 1$. Therefore, our cdf F is just

$$F(t) = \begin{cases} 0, & t < 0; \\ t^2, & 0 \leq t < 1; \\ 1, & t \geq 1. \end{cases}$$

Problem 6

(a) We want to know if there exists a constant c such that $P(X = k) = c/k^2$ is a legitimate pmf on the positive integers. Thus, is there a constant that satisfies the following equation (notice that we should get 1 if we sum over all possible values in the range of the random variable):

$$1 = \sum_{k=1}^{\infty} c/k^2 = c \sum_{k=1}^{\infty} 1/k^2.$$

Notice that the series on the right converges, so we see that there exists a c that solves the equation. In particular,

$$c = \left(\sum_{k=1}^{\infty} 1/k^2 \right)^{-1}.$$

(b) Notice that $\sum_{k=1}^{infty} c/k$ diverges for every $c > 0$. Also, c cannot be zero, because then we no longer have a density function. Therefore, there is no c that makes $P(X = k) = c/k$ a legitimate pmf.