Solutions to Homework 3, ISyE 2027 Spring 2006

Problem 1

A and B are not independent. Since A and B are mutually exclusive, $P(A \cap B) = 0$. However, $P(A) > 0$ and $P(B) > 0$ implies $P(A)P(B) > 0$, so A and B cannot be independent.

Problem 2

(a) The probability that both are at full capacity, given that neither line is shut down is just

$$P(\{(F,F)\}|\{(F,F),(F,P),(P,F),(P,P)\}) = \frac{P(\{(F,F)\})}{P(\{(F,F),(F,P),(P,F),(P,P)\})}$$

$$= \frac{0.19}{0.14 + 0.2 + 0.21 + 0.19} = 0.2568.$$

(b) The probability that at least one line is at full capacity conditional on neither line being shut down is just

$$P(\{(F,P),(F,S),(F,F),(P,F),(S,F)\}|\{(F,F),(F,P),(P,F),(P,P)\})$$

$$= \frac{P(\{(F,F),(F,P),(P,F)\})}{P(\{(F,F),(F,P),(P,F),(P,P)\})}$$

$$= \frac{0.20 + 0.19 + 0.21}{0.14 + 0.20 + 0.21 + 0.19} = 0.8108.$$

(c) The probability that one line is at full capacity, given that exactly one line is shut down is just

$$P(\{(F,P),(F,S),(F,F),(S,P),(S,F)\}|\{(S,P),(S,F),(F,S),(P,S)\})$$
$$= \frac{0.11}{0.06 + 0.07 + 0.06 + 0.05} = 0.4583.$$

(d) Finally, the probability that neither line is at full capacity, given that at least on line is operating at partial capacity is

$$P(\{(S,S),(S,P),(P,S)(P,P)\}|\{(P,P),(P,F),(P,S),(S,P),(F,P)\})$$
$$= \frac{0.06 + 0.07 + 0.14}{0.07 + 0.06 + 0.14 + 0.21 + 0.20} = 0.3971.$$

Problem 3

We know that A and B are independent.
(a) Notice that

$$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c).$$

(b) Similarly,

$$P(B \cap A^c) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = P(B)P(A^c).$$

(c) Finally, from part (a) we see that

$$P(B^c \cap A^c) = P(B^c) - P(B^c \cap A) = P(B^c) - P(B^c)P(A) = P(B^c)P(A^c).$$

Problem 4

(a) Our sample space is $\Omega = \{(b,b,b),(b,b,g),(b,g,b),(g,b,b),(g,g,b),(g,b,g),(b,g,g),(g,g,g)\}$, where for example (b,g,b) denotes the outcome where the first born child is a boy, the second is a girl, and the third is a boy. Our assumptions tell us that each outcome occurs with probability $1/8$, so $P(A) = 1/2$, $P(B) = 3/4$, and it is easy to see that $P(A \cap B) = 3/8$, which happens to equal $P(A)P(B)$. Hence, A and B are independent.
(b) When four children are present, you can verify yourself that $P(A) = 5/16$, $P(B) = 14/16$, but $P(A \cap B) = 4/16$, which is not equal to $P(A)P(B)$. Hence, A and B are not independent.

Problem 5

Since A and B are independent,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B).$$

Plugging in our values, we see that

$$0.64 = 0.4 + P(B) - (0.4)P(B)$$

so it follows that

$$P(B) = \frac{0.64 - 0.4}{1 - 0.4} = 0.4.$$

Problem 6

We see that

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

so it follows that $P(B \cap A) = P(B|A)P(A) = (1/4)(1/2) = 1/8)$. Similarly, note that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

so it follows that

$$P(B) = \frac{P(A \cap B)}{P(A|B)} = \frac{(1/8)}{(1/3)} = 3/8.$$

Finally,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/4 + 3/8 - 1/8 = 1/2.$$

Problem 7

Let L denote the event that the plane is late, and let R denote the event where it rains. We know that $P(L|R) = 0.8$, $P(L|R^c) = 0.15$, and $P(R) = 0.7$. Using the law of total probability, we see that

$$P(L) = P(L|R)P(R) + P(L|R^c)P(R^c) = (.8)(.7) + (.15)(.3) = 0.605.$$

Problem 8

Let C, L, and I be events denoting that the customer is conservative, liberal, or independent. Let V denote the event that the person voted. Notice that our goal is to compute $P(L|V)$. We know that $P(C) = 0.3$, $P(L) = 0.5$, and $P(I) = 0.2$. We also know that $P(V|C) = 2/3$, $P(V|L) = 4/5$, and $P(V|I) = 1/2$. By Bayes' Theorem,

$$P(L|V) = \frac{P(L)P(V|L)}{P(L)P(V|L) + P(C)P(V|C) + P(I)P(V|I)}$$

$$= \frac{(4/5)(.5)}{(4/5)(.5) + (2/3)(.3) + (1/2)(.2)} = 4/7.$$