Solutions to Homework 2, ISyE 2027 Spring 2006

Problem 1

We know that $P(A) = 0.6$, $P(A \cap B) = 0.4$, and $P(B - A) = 0.3$ (recall that $B - A = B \cap A^c$). Thus,

$$P(B) = P(B \cap A) + P(B - A) = 0.4 + 0.3 = 0.7.$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.7 - 0.4 = 0.9.$$

$$P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - 0.4 = 0.6.$$

$$P(B^c \cap A) = P(A) - P(A \cap B) = 0.6 - 0.4 = 0.2.$$

Problem 2

(a) By the multiplication rule, we see that the number of license plates that are possible is just $(26)^3(10)^3 = 17,576,000$.

(b) Let us first count the number of plates consisting of letters and numbers that are different. Notice that there are (ordering the positions from left to right) 26 choices for the first letter, 25 for the second, and 24 for the third. Likewise, there are 10 choices for the first number, 9 choices for the second and 8 choices for the third. Using the multiplication rule, we see that the number of plates that satisfy this criterion is $(26)(25)(24)(10)(9)(8) = 11{,}232{,}000$.

Therefore, the probability the three letters and the three numbers are all different is just 0.639.

Problem 3

Let A denote the event that the boys and the girls alternate. Furthermore, let A_G denote the event that the line is of the form $GBGBGB$ (in other words, a girl is at the head of the line, a boy is second, a girl third, a boy fourth, etc.). Similarly, let A_B denote the event that the line is of the form $BGBGBG$. Clearly,

$$P(A) = P(A_B) + P(A_G).$$

Let's first compute $P(A_B)$. Notice that there are 36 lines in the event A_B, since there are 3 ways to choose the leader, 3 ways to choose the second person, 2 ways to choose the third, 2 ways to choose the fourth, 1 way to choose the fifth and 1 way to choose the last person. Thus, $(3)(3)(2)(2)(1)(1) = 36$. This means that $P(A_B) = 36/6! = 36/720 = 1/20$, since there are 6! possible lines. Similarly, $P(A_G) = 1/20$, so it follows that

$$P(A) = 1/20 + 1/20 = 1/10.$$

Problem 4

The number of ways six people can sit in six seats in a line at a cinema is just $6! = 720$. The number of ways 6 people can sit around a dinner table eating pizza after the movie is $6!/6 = 120$ (accounting for rotations).

Problem 5

Let's count the number of possible lines that have Andrea sitting next to Scott, and have Scott sitting in one of seats 2,3,4,5 (I'm numbering seats from left to right). Notice that Scott has 4 chairs to choose from. Once Scott makes a choice, Andrea has 2 choices. One both of them have been seated, the next person has 4 choices, then the next has 3 choices, etc. Thus, the number of possible lines of this type is just $(4)(2)(4)(3)(2)(1) = 192$.

Now let's count the number of possible lines that have Andrea sitting next to Scott, and have Scott sitting in either seat 1 or seat 6. Now Scott has two choices. Once he makes a choice, Andrea only has 1 choice (since he's at the end of the row). Once both of them are seated, the next person has 4 choices, and so on. Hence, the number of possible linesn of this type is just $(2)(1)(4)(3)(2)(1) = 48$.

Combining both cases above tells us that the number of possible lines that have Andrea sitting next to Scott is just $192 + 48 = 240$.

Now let's assume that everyone's sitting around the table, and Andrea must sit next to Scott. Then Scott has 6 choices available. Once he makes a choice, Andrea has 2 choices, the next person has 4 choices, and so on. Hence, the number of possible ways is just $(6)(2)(4)(3)(2)(1)/6 = 48$.

Now assume that Andrea and Scott cannot sit next to each other. Using the same kind of logic, we see that the number of ways they can sit at the cinema is $(4)(3)(4)(3)(2)(1) + (2)(4)(4)(3)(2)(1) = 288 + 192 = 480$. Also, the number of ways they can sit around the table is just $(6)(3)(4)(3)(2)(1)/6 = 72$.

Problem 6

Here we are given a collection of 60 items, and a quality inspector selects a sample of 12 items. Notice that order does not matter here (we only care about what's in the sample), so our sample space will just consist of all possible subsets of size 12 (all sample possibilities). Our assumptions tell us that each sample is equally likely to occur.

(a) The probability that the sample only contains items that have either excellent or good quality is just

$$\frac{\binom{43}{12}}{\binom{60}{12}}.$$

Let's explain how we arrived at this answer. Obviously there are $\binom{60}{12}$ possible samples. Now, the number of samples that contain only excellent or good items is just the number of subsets of size 12 that we can form using only excellent or good items, which is just $\binom{43}{12}$.

(b) The probability that the sample contains three items of excellent quality, three items of good quality, three items of poor quality and three defective items is just

$$\frac{\binom{18}{3}\binom{25}{3}\binom{12}{3}\binom{5}{3}}{\binom{60}{12}}.$$

We want to count all subsets that have three items of each type. Notice that the number of ways we can choose 3 excellent items to go in our sample is just $\binom{18}{3}$. Once we've made that choice, the number of ways we can choose 3 good items is just $\binom{25}{3}$. Once we've chosen our excellent and good items, the number of ways we can choose 3 poor items is just $\binom{12}{3}$, and finally the number of ways to choose 3 defective items is just $\binom{5}{3}$. Thus, the number of ways we can have three items of each type in our sample can be found using the multiplication rule.

Problem 7

To compute the probability that she can solve at least 5 problems of the test, it suffices to compute the probability that she can solve exactly 5 problems, and the probability that she can solve exactly six problems.

Using logic similar to that found in Problem 6, we see that the probability she can solve 5 problems on the test is just

$$\frac{\binom{9}{5}\binom{3}{1}}{\binom{12}{6}}.$$

Also, the probability she can solve 6 problems on the test is just

$$\frac{\binom{9}{6}}{\binom{12}{6}}.$$

Therefore, the probability she can solve at least 5 problems on the test is

$$\frac{\binom{9}{5}\binom{3}{1} + \binom{9}{6}}{\binom{12}{6}}$$