Solutions to Homework 10, ISyE 2027 Spring 2006

Problem 1

We know that X is a continuous random variable with pdf

$$f(x) = \begin{cases} 1 - |x|, & -1 \le x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 + x, & -1 \le x < 0; \\ 1 - x, & 0 \le x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the Moment Generating function of X is just

$$M_X(t) = E[e^{tX}] = \int_{-1}^{0} (1 + x)e^{tx}dx + \int_{0}^{1} (1 - x)e^{tx}dx$$

$$= \int_{-1}^{1} e^{tx}dx + \int_{-1}^{0} xe^{tx}dx - \int_{0}^{1} xe^{tx}dx$$

$$= \frac{e^t}{t} - \frac{e^{-t}}{t} + \frac{e^{-t}}{t} - \frac{1}{t}\int_{-1}^{0} e^{tx}dx - \frac{e^t}{t} + \frac{1}{t}\int_{0}^{1} e^{tx}dx$$

$$= \frac{e^t + e^{-t} - 2}{t^2}.$$

Problem 2

(a) Suppose N is a geometric random variable with parameter p. Then

$$M_N(t) = E[e^{tX}] = \sum_{k=1}^{\infty} e^{tk} P(N = k) = \sum_{k=1}^{\infty} e^{tk} p(1 - p)^{k-1}$$

$$= pe^t \sum_{k=1}^{\infty} ((1 - p)e^t)^{k-1}.$$

Notice that if $t < -\ln(1 - p)$, then $(1-p)e^t < 1$. If we impose this condition on t, then the above series is a geometric series, so it follows that

$$M_N(t) = \frac{pe^t}{1 - (1 - p)e^t}.$$

Remember that we only require that the moment generating function exists on a neighborhood of zero (i.e. there exists an $h > 0$ such that $M_N(t) < \infty$ for $t < h$). It doesn't have to be finite for every t.
(b) Using the quotient rule for derivatives, we see that

$$M_N^{'}(t) = \frac{(1-(1-p)e^t)pe^t + p(1-p)e^{2t}}{(1-(1-p)e^t)^2},$$

so from class we know that

$$E[N] = M_N^{'}(0) = \frac{p^2 + p - p^2}{p^2} = \frac{1}{p}.$$

(c) Suppose B is a negative binomial random variable with parameters r and p. Notice that if X_1, X_2, ..., X_r are iid Geometric(p) random variables, then

$$\sum_{k=1}^{r} X_k$$

is also negative binomial with parameters r and p. Therefore,

$$M_B(t) = M_{\sum_{k=1}^{r} X_k}(t) = M_{X_1}(t) M_{X_2}(t) ... M_{X_r}(t)$$

$$= \left(\frac{pe^t}{1-(1-p)e^t}\right)^r.$$

Problem 3

The moment generating function of Z^2 is just

$$M_{Z^2}(t) = E[e^{tZ^2}] = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2(\frac{1}{2}-t)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}(1-2t)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1-2t)^{-1}}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1-2t)^{-1}}} dx$$

$$= (1-2t)^{-1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-2t)^{-1}}} e^{-\frac{x^2}{2(1-2t)^{-1}}} dx$$

but now what we're integrating is just the density of a normal random variable with mean 0 and variance $(1-2t)^{-1}$ (assuming of course that $t < 1/2$), so this expression is just

$$= (1-2t)^{-1/2} = \left(\frac{(1/2)}{(1/2)-t} \right)^{1/2}.$$

which is the moment generating function of a gamma random variable with parameters $\alpha = 1/2$, and $\lambda = 1/2$.

Problem 4

We know that

$$M_X(t) = .5 + (.2)e^{2t} + (.1)e^{3t} + (.2)e^{5t}.$$

(a)

$$E[X] = M_X^{'}(0) = (.4) + (.3) + 1 = 1.7.$$

(b)

$$E[X^2] = M_X^{''}(0) = (.8) + (.9) + 5 = 6.7$$

so

$$Var(X) = 6.1 - (1.7)^2 = 2.81.$$

(c)

$$E[X^4] = M_X^{(4)}(0) = 3.2 + 8.1 + 125 = 136.3.$$

Note: In Problems 5, 6, and 7, Z is a standard normal random variable

Problem 5

Let $X =$ the number of hits out of 168 at-bats. If we let X_k equal 1 if he gets a hit during the at-bat k, and 0 otherwise, then (assuming that performances during at-bats are independent of one another)

$$P(X \leq 42) = P(X \leq 42.5) = P(\sum_{k=1}^{168} X_k \leq 42.5)$$

$$= P\left(\frac{\sum_{k=1}^{168} X_k - (168)(.3)}{\sqrt{(168)(.3)(.7)}} \leq \frac{42.5 - (168)(.3)}{\sqrt{(168)(.3)(.7)}}\right)$$

(note the use of the continuity correction). Using the CLT, we have that this is

$$\approx P(Z \leq -1.33) = 0.0918.$$

Problem 6

Let W_k denote the weight of person k. Then

$$P(\sum_{k=1}^{36} W_k > 6500) = P\left(\frac{\sum_{k=1}^{36} W_k - 36(175)}{16(6)} > 2.0833\right)$$

$$\approx P(Z > 2.083) = P(Z < -2.083) = 0.0188.$$

Problem 7

Again

$$P(B \geq 23) = P(B \geq 22.5) = P\left(\frac{B - 25(.8)}{\sqrt{25(.8)(.2)}} \geq \frac{22.5 - 25(.8)}{\sqrt{25(.8)(.2)}}\right)$$

$$\approx P(Z \geq 1.25) = P(Z \leq -1.25) = 0.1056.$$