We study the i.i.d. online bipartite matching problem, a dynamic version of the classical model where one side of the bipartition is fixed and known in advance, while nodes from the other side appear one at a time as i.i.d. realizations of an underlying distribution, and must immediately be matched or discarded. We consider time-indexed relaxations of the set of achievable matching probabilities, introduce complete subgraph inequalities, show how they theoretically dominate inequalities from a lower-dimensional relaxation presented in previous work, and discuss when they are facet-defining. We finally present a computational study to demonstrate the empirical quality of the new relaxations and the heuristic policies they imply.
two different heuristics: A one suggested matching heuristic matches only edges from a specific matching computed beforehand; this algorithm achieves a $1 - \frac{1}{e}$ approximation ratio, equal to the best-possible competitive ratio for the adversarial case. The two suggested matchings heuristic improves on this ratio, using two different matchings instead of one. Several results have since improved this approximation guarantee with other heuristic policies, e.g. [2, 18]. The best currently known approximation ratio is from the heuristic in [12], which uses a max-flow relaxation to create a randomized policy. All of these results pertain to the i.i.d. arrival model with max-cardinality objective; for other results and variants see e.g. [9, 11, 17] and the survey [19].

Most of this previous work is focused on developing heuristics with constant-factor approximation ratios. We instead approach OBM from an explicitly polyhedral point of view, with the primary goal of constructing linear programming (LP) relaxations that yield good upper bounds. While some of the previously mentioned algorithms are constructed using simple LP relaxations, e.g. [7, 12], we use an exact dynamic programming (DP) formulation and a corresponding polytope of achievable probabilities as starting points. This focus on the achievable region has seen application in other contexts, such as queueing and multi-armed bandits [3, 5], but its only previous application in OBM is [22].

1.1 Summary of Contributions

Within this context, we consider the following to be our main contributions. Our first contribution is a polyhedral study of the convex hull of time-indexed achievable probabilities. Previous work [22] studies coarser, “static” relaxations, in which probabilities do not depend on the stage, and our perspective allows for a tighter analysis. By using time-indexed probabilities, we are also able to model the weighted case, where weights can even vary over time; this in turn implies that our results do not depend on the structure of the underlying graph, as we can capture all structural information in the objective. For this polyhedron of time-indexed probabilities, we present a general family of valid inequalities constructed by choosing complete bipartite subgraphs, and discuss various cases in which they are facet-defining.

To verify the quality of our newly introduced inequalities, as a second contribution we compare them theoretically and empirically with the best-performing bounds from [22]. Our inequalities in fact imply this paper’s right-star inequalities, which produce the best empirical bounds. Furthermore, our computational experiments show that our inequalities strictly outperform the right-star family, often by a significant margin; we are able to reduce the gap in absolute terms by 4% to 5% on average in our tested instances. Our empirical results also show that in many instances, particularly in large dense graphs, our inequalities give the tightest known upper bound, beating even the sample mean of the off-line max-weight matching. Finally, we show that a heuristic policy derived from our bound also clearly outperforms the best policy from [22] and is near-optimal in many cases.

In the remainder of the paper, Section 2 formulates the problem, and summarizes pertinent previous work. Section 3 introduces our inequalities and gives our theoretical results. Section 4 outlines our computational study and its results. Finally, Section 5 presents conclusions and discusses possible future work. An Appendix has mathematical proofs not included in the body of the paper.

2 Model Description and Preliminaries

The OBM model is formulated using two finite disjoint sets $N$ and $V$, with the process occurring dynamically in the following way. The right-hand node set $V$, with $|V| = m$, is known and given ahead of time. The left-hand set $N$ with $|N| = n$ represents different node types that may appear, but we do not know which ones will appear and how often. We know only that $T$ left-hand nodes in total will appear sequentially, each one drawn independently from the uniform distribution over node types $N$. That is, at each epoch a node from one of the types
in $N$ appears with probability $1/n$ and must be immediately (and irrevocably) matched to a remaining available node in $V$ or discarded; two or more nodes from the same type may appear throughout the process, each treated as a separate copy. Matching $i \in N$ to $j \in V$ in stage $t$ yields a (known) reward or weight $w^t_{ij}$, and the objective is to maximize the expected weight of the matching. Following convention from previous literature and the motivating application of search engine advertisement, we call $i \in N$ an impression, and each $j \in V$ an ad.

By considering time-indexed weights $w^t_{ij}$, we generalize much of the existing literature and can avoid dealing with specific graph structure. In particular, we may assume that the process occurs in a complete bipartite graph, i.e. every node type in $N$ is connected or compatible with every node in $V$: non-existent edges simply get weight zero.

Moreover, we can assume $m = n = T$ without loss of generality. Indeed, if $m < T$ we add dummy nodes to $V$ and assign zero weight to all corresponding edges. Similarly, if $n > m = T$ we increase the number of stages and give zero weight to all edges in the new stages. If $n > m = T$, we again add dummy nodes and stages. Finally, if $n < m = T$ we make $\kappa$ copies of every node type in $N$ (and the corresponding edges) for the smallest $\kappa$ with $\kappa n \geq m$, then proceed as before. To ease notation, in the remainder of the paper we write $n$ for $m$ and $T$, but we use the indices $i$ for impressions, $j$ for ads, and $t$ for stages. We use the shorthand $[n] := \{1, \ldots, n\}$, and identify singleton sets with their unique element.

### 2.1 DP and LP Formulations

Let $\eta$ be the random variable with uniform distribution over $N$. We count stages down from $n$, meaning stage $t$ occurs when $t$ decision epochs (including the current one) remain in the process. We can now give a DP formulation for this OBM model. Let $v^*_t(i,S)$ denote the optimal expected value given that $i \in N$ appears in stage $t$ when the set of ads $S \subseteq V$ is available. Then, for all $t = 1, \ldots, n$, $i \in N$ and $S \subseteq V$,

$$v^*_t(i,S) = \max \left\{ \max_{j \in S} \left[ w^t_{ij} + \mathbb{E}_\eta[v^*_{t-1}(\eta,S\setminus j)] \right], \mathbb{E}_\eta[v^*_{t-1}(\eta,S)] \right\}$$

where $v^*_0(\cdot, \cdot)$ is identically zero, and the optimal expected value of the model is given by $\mathbb{E}_\eta[v^*_n(\eta,V)] = 1/n \sum_{i \in N} v^*_n(i,V)$. The first term in this recursion corresponds to matching $i$ with one of the remaining ads $j \in S$: the second corresponds to discarding $i$. As with any DP, the optimal value function $v^*$ induces an optimal policy: At any state $(t, i, S)$, we choose an action that attains the maximum in (1).

Using a standard reformulation (see e.g. [20]), we can capture the recursion (1) with the linear program

$$\begin{align*}
\min_{v \geq 0} & \quad \mathbb{E}_\eta[v_n(\eta,V)] \\
\text{s.t.} & \quad v_t(i, S \cup j) - \mathbb{E}_\eta[v_{t-1}(\eta,S)] \geq w^t_{ij}, \quad t \in [n], \ i \in N, \ j \in V, \ S \subseteq V \setminus j \\
& \quad v_t(i, S) - \mathbb{E}_\eta[v_{t-1}(\eta,S)] \geq 0, \quad t \in [n], \ i \in N, \ S \subseteq V.
\end{align*}$$

The value function $v^*$ defined in (1) is optimal for (2). Moreover, this LP is a strong dual for OBM: any feasible $v$ has an objective greater than or equal to $\mathbb{E}_\eta[v^*_n(\eta,V)]$. The dual of (2) is a primal formulation where any feasible solution encodes a feasible policy and its probability of
achievable vector of matching probabilities that is where \( z \) policy makes a particular match between \( i \) difficult to analyze directly. However, we can equivalently consider the probability that a feasible action chooses to match impression \( i \) to be a relaxation of \( Q \) via \( z \) projected polyhedron in the space of \( z \) where \( t \) feasible region of (3). Specifically, assuming edge weights are static across stages, \( w \) Most previous results concerning relaxations for OBM use a lower-dimensional projection of the \( Q \) yields a valid upper bound; this is our main goal.

As with its dual, (3) has exponentially many variables and constraints, and is therefore difficult to analyze directly. However, we can equivalently consider the probability that a feasible policy makes a particular match between \( i \) and \( j \) in stage \( t \) without tracking the other remaining ads \( S \subseteq V \setminus j \); this corresponds to optimizing over a projection of the feasible region of (3),

\[
\max \left\{ \sum_{i \in N} \sum_{j \in V} \sum_{t \in [n]} w_{ij}^t z_{ij} : \exists (x, y) \geq 0 \text{ satisfying } (3b)-(3d) \text{ with } z_{ij}^t = \sum_{S \subseteq V \setminus j} x_{ij}^t \right\},
\]

where \( z_{ij}^t \) is the probability that impression \( i \) is matched to ad \( j \) in stage \( t \). Any such \( z \) is a vector of matching probabilities that is achievable by at least one feasible policy. Let \( Q \) denote this projected polyhedron in the space of \( z_{ij}^t \) variables, and note that \( Q \) is full-dimensional in \( \mathbb{R}^{n^3} \). Optimizing over \( Q \) is as difficult as solving the original DP formulation (1), but optimizing over any relaxation of \( Q \) yields a valid upper bound; this is our main goal.

### 2.2 Relevant Previous Work

Most previous results concerning relaxations for OBM use a lower-dimensional projection of the feasible region of (3). Specifically, assuming edge weights are static across stages, \( w_{ij} = w_{ij} \) for \( t \in [n] \), consider

\[
\max \left\{ \sum_{i \in N} \sum_{j \in V} w_{ij} z_{ij} : \exists (x, y) \geq 0 \text{ satisfying } (3b)-(3d) \text{ with } z_{ij} = \sum_{t \in [n]} \sum_{S \subseteq V \setminus j} x_{ij}^t \right\},
\]

where \( z_{ij} \) is the probability that impression \( i \) is ever matched to ad \( j \). Let \( Q' \) denote this projected polyhedron in the space of \( z_{ij} \) variables, and observe that \( Q' \) is also a projection of \( Q \) via \( z_{ij} = \sum_{t \in [n]} z_{ij}^t \). The following max flow (or deterministic bipartite matching) LP is known to be a relaxation of \( Q' \) and has been used to study \( Q' \) in several works starting with [7]:

\[
\max_{z \geq 0} \quad \sum_{i \in N} \sum_{j \in V} w_{ij} z_{ij} \\
\text{s.t.} \quad \sum_{j \in V} z_{ij} \leq T/n = 1, \quad i \in N \\
\quad \sum_{i \in N} z_{ij} \leq 1, \quad j \in V.
\]
In this relaxation, constraints (4b) limit the expected number of times an impression type can be matched to $T/n = 1$, the expected number of times it will appear, while (4c) state that each ad is matched at most once.

To our knowledge, the only past work that specifically focuses on polyhedral relaxations of $Q'$ is [22], which presents several classes of valid inequalities, including the right-star inequalities, which yield the best empirical bounds when added to (4). Although exponential in number, these inequalities can be separated over in polynomial time by a simple greedy algorithm. We use the bound given by (4) with (5) as a theoretical and empirical benchmark to test our new relaxations.

3 Time-Indexed Relaxations

We introduce various classes of valid inequalities for $Q$ and study their facial dimension. These inequalities always include variables corresponding to complete bipartite subgraphs; therefore, to ease notation we define

$$Z_{i,t} = \sum_{j \in J} \sum_{i \in I} z_{ij}, \quad I \subseteq N, J \subseteq V.$$ 

We begin by presenting a simple inequality class to motivate our approach. For an impression $i \in N$, the probability of matching $i$ in each stage $t \in [n]$ is at most $1/n$; this corresponds to

$$Z_{i,t} \leq 1/n, \quad i \in N, t \in [n].$$

Note that by summing these constraints over all $t$ for a fixed $i$, we obtain (4b).

**Proposition 1.** Constraints (6) are facet-defining for the polyhedron of achievable probabilities $Q$.

**Proof.** Fix $i \in N$ and $t \in [n]$. We use $e_{k,j}^t \in \mathbb{R}^n$ to denote the canonical vector, i.e., a vector with a one in the coordinate $(k, j, \tau)$ and zero elsewhere, indicating that we match impression $k$ with ad $j$ in stage $\tau$. We construct the following $n^3$ affinely independent points corresponding to policies that satisfy (6) with equality:

- Policy for $(i, j, t)$ with $j \in V$: If $i$ appears in stage $t$, which happens with probability $1/n$, we match it with $j$. This corresponds to the point $\frac{1}{n} e_{k,j}^t$.
- Policy for $(k, j, \tau)$ with $j \in V$, $\tau \neq t$, and $k \in N$: If $k$ appears in stage $\tau$ (with probability $1/n$), we match it to $j$. Then, if $i$ appears in stage $t$ with probability $1/n$, we match it to some $\ell \in V$, $\ell \neq j$, so we have the point $\frac{1}{n} e_{k,j}^t + \frac{1}{n} e_{\ell,i}^t$.
- Policy for $(k, j, t)$ with $j \in V$, and $k \neq i$: If $k$ appears in stage $t$ (with probability $1/n$), we match it to $j$. On the other hand, if $i$ appears in stage $t$ with probability $1/n$, we match it to some $\ell \in V$, $\ell \neq j$, so we have the point $\frac{1}{n} e_{k,j}^t + \frac{1}{n} e_{i,\ell}^t$.

These points are linearly independent, which implies they are affinely independent. \hfill $\square$

We now introduce our general inequality family. Fix a set of ads $J \subseteq V$ and a family of impression sets $I_t \subseteq N$, $t \in [n]$. For any vector $\alpha \in \mathbb{R}^n_+$, we have a valid inequality for $Q$ of the form

$$\sum_{t=1}^n \alpha_t Z_{i,t} \leq R(\alpha, (I_t), J),$$

where $R(\alpha, (I_t), J)$ defines the maximum of the left-hand side over $Q$. As one example, inequalities (6) are a special case of (7) where $J = V$, $I_t = \{i\}$, $\alpha_t = 1$, and $I_\tau = \emptyset$, $\alpha_\tau = 0$ for $\tau \neq t$. 


Proof. Define variables \( p_t \in \{0,1\} \) to indicate whether a node from \( I_t \) appears in stage \( t \) or not, and denote by \( d \in \{0,\ldots,|J|\} \) the number of remaining nodes from \( J \). Given this, we can state a DP recursion using the value function \( R(t,d,p_t) \), the expected value in stage \( t \) when \( d \) nodes from \( J \) are available and \( p_t \) has occurred. For example, if only one stage remains, \( d \) nodes from \( J \) are available, and no element of \( I_1 \) appears, \( R(1,d,0) = 0 \) since we cannot match any node in \( I_1 \). Conversely, \( R(1,d,1) = \alpha_1 \min\{1,d\} \), since we can match a node and obtain value \( \alpha_1 \) as long as at least one element of \( J \) remains.

In general, if \( d \) nodes are available in stage \( t \) and no node from \( I_t \) appears (\( p_t = 0 \)), then the expected value \( R(t,d,0) \) can be computed recursively by conditioning on terms from stage \( t-1 \):

\[
R(t,d,0) = \frac{n - |I_t|}{n} R(t-1,d,0) + \frac{|I_t|}{n} R(t-1,d,1).
\]

On the other hand, to compute \( R(t,d,1) \) we choose the maximum between discarding or matching, with value

\[
R(t,d,1) = \max\{R(t-1,d,0),\alpha_1 + R(t-1,d-1,0)\}
\]

Finally, the value of the right-hand side is

\[
R(\alpha,(I_t),J) = \frac{n - |I_n|}{n} R(n,|J|,0) + \frac{|I_n|}{n} R(n,|J|,1).
\]

The number of states is \( n \times |J| \times 2 = O(n^2) \) and the number of operations to calculate a state’s value is constant, so the entire recursion takes \( O(n^2) \) time.

In the remainder of this section, we study particular cases of inequalities (7). We construct them intuitively using probabilistic arguments, but their right-hand sides can also be calculated directly using the DP from Proposition 2.

As a first example, let \( i \in \mathcal{N} \), \( j \in \mathcal{V} \) and \( t \in [n-1] \). Matching \( i \) to \( j \) in stage \( t \) implies the intersection of two independent events. First, \( j \) is not matched in any previous stage \([t+1,n]\), and second, \( i \) appears in stage \( t \). In terms of probability this means

\[
\mathbb{P}(\text{match } i \text{ with } j \text{ in } t) \leq \frac{1}{n} (1 - \mathbb{P}(\text{match } j \text{ in } [t+1,n])),
\]

which is equivalent to

\[
\mathbb{P}(\text{match } j \text{ in stages } [t+1,n]) + n \mathbb{P}(\text{match } i \text{ with } j \text{ in } t) \leq 1.
\]

The previous expression is equivalent to

\[
\sum_{\tau=t+1}^{n} Z_{N,j}^{\tau} + nz_{i,j}^{t} \leq 1 \quad \forall \ i \in \mathcal{N}, \ j \in \mathcal{V}, \ t \in [n].
\]

Inequality family (8) corresponds to a particular case of (7), with \( |J| = 1 \), \( I_t = N \) for \( \tau \in [t+1,n] \), \( |I_t| = 1 \), \( I_{\tau} = \emptyset \) for \( \tau \leq t-1 \), \( \alpha_\tau = 1 \) for \( \tau \in [t+1,n] \), \( \alpha_t = n \), and \( \alpha_\tau = 0 \) for \( \tau \leq t-1 \). Furthermore, for a fixed \( j \in \mathcal{V} \) and \( t = 1 \), by summing the inequalities over all \( i \in \mathcal{N} \) we obtain (4c).

Proposition 3. Constraints (8) are facet-defining for the polyhedron of achievable probabilities \( Q \) when \( t \leq n-1 \).

Proof. Fix \( i \in \mathcal{N}, \ j \in \mathcal{V}, \ t \in [n-1] \). We construct the following \( n^3 \) affinely independent points corresponding to policies that satisfy (8) with equality:
1. Policy for \((i,j,t)\): if \(i\) appears in stage \(t\), then match it to \(j\) with probability \(1/n\). This corresponds to the point \(\frac{1}{n}e_{i,j}^t\).

2. Policy for \((k,j,\tau)\) with any \(k \in N\), and any \(\tau \in [t+1,n]\): If \(k\) appears in stage \(\tau\), match it to \(j\) with probability \(1/n\), but if \(k\) does not appear and \(i\) appears in stage \(t\), we match \(i\) to \(j\) with probability \(\frac{1}{n} (1 - \frac{1}{n})^t\). This corresponds to the point \(\frac{1}{n}e_{k,j}^t + \frac{1}{n} (1 - \frac{1}{n}) e_{i,j}^t\). As we chose any \(k\) and any \(\tau\), we have \(n(t-1)\) points.

So far, we only have \(n(n-t)+1\) points. For the remaining points, we can use modifications of policy 1 above.

- Policy for \((k,j,\tau)\) with any \(k \in N\) and \(\tau \leq t-1\): If \(i\) appears in stage \(t\) with probability \(1/n\), then match it with \(j\); if \(i\) does not appear (with probability \(1-1/n\)), and if \(k\) appears in stage \(\tau\) (with probability \(1/n\)), then match it with \(j\). This corresponds to \(\frac{1}{n}e_{i,j}^t + \frac{1}{n} (1 - \frac{1}{n}) e_{k,j}^\tau\). As we chose any \(k\), and any \(\tau \leq t-1\), we have \(n(t-1)\) points.

- Policy for \((k,\ell,\tau)\) with any \(k \in V\), \(\ell \in V\) such that \(\ell \neq j\), and \(\tau \in [n]\): if \(i\) appears in stage \(t\) with probability \(1/n\), then match it with \(j\); if \(k\) appears in stage \(\tau\) (with probability \(1/n\)), then match it with \(\ell\). This corresponds to \(\frac{1}{n}e_{i,j}^t + \frac{1}{n} e_{k,\ell}^\tau\). In total, this yields \(n(n-1)n\) points.

- Policy for \((k,j,t)\) with \(k \in V\) such that \(k \neq i\): if \(i\) appears in stage \(t\) with probability \(1/n\), then match it with \(j\); if \(k\) appears in stage \(t\) (with probability \(1/n\)), then match it with \(j\). This corresponds to \(\frac{1}{n}e_{i,j}^t + \frac{1}{n} e_{k,j}^t\). In this family, we have \(n-1\) points.

If we order these points in a suitable way, they form the columns of a block matrix

\[
A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},
\]

where \(A_1\) is upper triangular and \(A_3\) is a diagonal matrix. \(A_1\) is formed by the first \(n(n-t)+1\) points from policy 1 and the policies of item 2, while \(A_2\) and \(A_3\) are given by the remaining points. All diagonal entries of \(A_1\) and \(A_3\) are positive, implying that \(A\) has positive determinant. This shows that the points previously described are linearly independent, completing the proof. \(\square\)

We next compare the inequalities we have introduced so far to the known results for the lower-dimensional polyhedron \(Q'\) of achievable probabilities that are not time-indexed, which we detail in Section 2. Recall that \(Q'\) is a projection of \(Q\) obtained by aggregating variables \(z_{ij}^t\) over all stages, \(z_{ij} = \sum_{t \in [n]} z_{ij}^t\). We already indicated how inequality families (4b) and (4c) are obtained by aggregating variables \(z_{ij}^t\) over all stages, \(z_{ij} = \sum_{t \in [n]} z_{ij}^t\). We already indicated how inequality families (4b) and (4c) are obtained by aggregating variables \(z_{ij}^t\) over all stages, \(z_{ij} = \sum_{t \in [n]} z_{ij}^t\). We already indicated how inequality families (4b) and (4c) are obtained by aggregating variables \(z_{ij}^t\) over all stages, \(z_{ij} = \sum_{t \in [n]} z_{ij}^t\).

**Theorem 4.** Inequalities (8) imply the right-star inequalities (5).

**Proof.** Fix \(j \in V\) and \(I \subseteq N\). First, for \(t = n\) (8) is simply \(n z_{ij}^n \leq 1\) (it is also a weakened version of (6)), and summing over \(I\) we get \(n \sum_{i \in I} z_{ij}^n \leq |I|\). For \(t \leq n-1\), if we sum over \(i \in I\) in (8) we get

\[
|I| \sum_{\tau \in [t+1,n]} \sum_{k \in N} z_{k,j}^\tau + n \sum_{i \in I} z_{i,j}^t \leq |I|, \quad \forall t \in [n-1],
\]

and since \(\sum_{i \in I} z_{i,j}^t \leq \sum_{k \in N} z_{k,j}^\tau\), we have

\[
|I| \sum_{\tau \in [t+1,n]} \sum_{i \in I} z_{i,j}^\tau + n \sum_{i \in I} z_{i,j}^t \leq |I|, \quad \forall t \in [n-1].
\]

Then, multiply each inequality for \(t \in [n-1]\) by \(\frac{1}{n} (1 - \frac{|I|}{n})^{t-1}\), and add all of them (including the one for \(t = n\)); the resulting coefficient for each \(z_{ij}^t\) is

\[
\left(1 - \frac{|I|}{n}\right)^{t-1} + \sum_{\tau \leq t-1} \frac{|I|}{n} \left(1 - \frac{|I|}{n}\right)^{\tau-1} = 1.
\]
We thus obtain \( \sum_{t \in [n]} \sum_{i \in I} z_{ij}^t = \sum_{t \in I} z_{ij} \) in the left-hand side. In the right-hand side, we get

\[
\frac{|I|}{n} \sum_{t=1}^{n} \left( 1 - \frac{|I|}{n} \right)^{t-1} = 1 - \left( 1 - \frac{|I|}{n} \right)^n.
\]

This result shows that inequalities (6) and (8) yield an upper bound that theoretically dominates the bound given by LP (4) with additional inequalities (5); this bound performs the best in computational experiments among all the bounds tested in [22]. In terms of dimension, the LP given by (6) and (8) with non-negativity constraints has \( O(n^3) \) inequalities in \( \mathbb{R}^n \), while (4) with (5) has exponentially many inequalities in \( \mathbb{R}^{n^2} \).

Inequalities (8) correspond to a particular case of (7), when the fixed set of ads \( J \) has one element. We can apply a similar idea to a subset of any size; take the next simplest case of (7), a set of size two, say \( J = \{j_1, j_2\} \). Consider also two impressions \( i_1, i_2 \in N \), where we may have \( i_1 = i_2 \). In terms of probability, the event of matching \( i_2 \) with \( j_1 \) or \( j_2 \) in stage \( t \) implies \( i_2 \) must appear in stage \( t \) with probability \( 1/n \) and either of two events happens: First, neither \( j_1 \) nor \( j_2 \) are matched in stages \([t + 2, n]\), and then \( i_1 \) appears in stage \( t + 1 \) with probability \( 1/n \) (and can be matched to one of the ads or not); and second, \( j_1 \) or \( j_2 \) (but not both) are matched in stages \([t + 2, n]\), and \( i_1 \) is not matched to \( j_1 \) nor \( j_2 \) in stage \( t + 1 \) (this includes the case of another impression being matched to one of them). Since matching \( j_1 \) or \( j_2 \) in \( t \) are mutually exclusive events, we have the inequality

\[
\mathbb{P}(\text{match } i_2 \text{ with } j_1 \text{ or } j_2 \text{ in } t) \leq \frac{1}{n} \left[ \frac{1}{n^2} (1 - \mathbb{P}(\text{match } j_1 \text{ or } j_2 \text{ in } [t + 2, n])) + (1 - \mathbb{P}(\text{match } i_1 \text{ with } j_1 \text{ or } j_2 \text{ in } t + 1)) \right].
\]

In terms of variables \( z \), this is equivalent to

\[
\sum_{\tau \in [t+2, n]} Z_{N,J}^\tau + nZ_{i_i+t,J}^{t+1} + n^2 Z_{i_t,J}^t \leq 1 + n \quad \forall i_t, i_{t+1} \in N, \; J \subseteq V, \; |J| = 2, \; t \in [n - 2]. \tag{9}
\]

This probabilistic argument can be generalized for any set \( J \subseteq V \) with \( |J| = h \in [n - 1] \) and any \( t \leq n - h \). Let \((i_1, \ldots, i_h)\) be a sequence of nodes in \( N \); the general constraint corresponds to

\[
\mathbb{P}(\text{match } i_h \text{ to some } j \in J \text{ in } t) \\
\leq \frac{1}{n} \left[ \frac{1}{n^{h-1}} (1 - \mathbb{P}(\text{match } j_1 \text{ or } j_2 \text{ or } \ldots \text{ or } j_h \text{ in } [t + h, n])) \right. \\
+ \frac{1}{n^{h-2}} (1 - \mathbb{P}(\text{match } i_1 \text{ to some } j \in J \text{ in } t + h - 1)) \\
+ \frac{1}{n^{h-3}} (1 - \mathbb{P}(\text{match } i_2 \text{ to some } j \in J \text{ in } t + h - 2)) \\
+ \cdots + (1 - \mathbb{P}(\text{match } i_{h-1} \text{ to some } j \in J \text{ in } t + 1)) \right].
\]

Therefore, we can give a general expression for this particular subclass of inequalities (7):

\[
\sum_{\tau = t+h}^n Z_{N,J}^\tau + \sum_{\tau = t}^{t+h-1} n^{t+h-\tau} Z_{i_t,J}^\tau \leq 1 + \sum_{\tau = 1}^{h-1} n^\tau, 
\tag{10}
\]

\( \forall J \subseteq V, \; |J| = h \in [n - 1], \; t \in [n - h], \; i_t, \ldots, i_{t+h-1} \in N. \)

**Theorem 5.** Constraints (10) are facet-defining for \( Q \).

The proof of this theorem can be found in the Appendix.
So far we have only considered either \( I_r = N \) or \(|I_r| = 1\) within inequalities (7). We next propose a generalization for other sets \( I \). Consider the case \( J = \{ j \} \), and any subset \( I \subseteq N \); suppose we naively apply the same argument behind inequality (8). Matching an element of \( I \) with \( j \) in stage \( t \) implies the intersection of two independent events: First, \( j \) is not matched in stages \([t + 1, n]\), and second, some element in \( I \) appears in stage \( t \). In probabilistic terms,

\[
P(\text{match any element in } I \text{ with } j \text{ in } t) \leq \frac{|I|}{n} (1 - P(\text{match } j \text{ in } [t + 1, n])),
\]

which is equivalent to

\[
|I| \sum_{\tau = t+1}^{n} Z_{N,j}^{\tau} + nZ_{t,j}^{t} \leq |I|.
\]

However, this inequality is made redundant by (8), because we can sum over \( i \in I \) for the same fixed \( t \) to get it.

Consider instead \( J = \{ j_1, j_2 \} \), any \( I_1 \subseteq N \) with \(|I_1| \geq 2\), and another impression \( i_2 \in N \); we apply the same argument used for (9), but substituting \( I_1 \) for the single impression \( i_1 \). Matching \( i_2 \) with \( j_1 \) or \( j_2 \) in stage \( t \) implies \( i_2 \) appears in stage \( t \) with probability \( 1/n \), and either of two previous events happens: First, neither \( j_1 \) nor \( j_2 \) are matched in stages \([t + 2, n]\), and then any element in \( I_1 \) appears in stage \( t + 1 \) with probability \(|I_1|/n\) (and is matched to one of the ads or not); and second, one of \( j_1 \) or \( j_2 \) is matched in stages \([t + 2, n]\), and no element from \( I_1 \) is matched to \( j_2 \) nor \( j_2 \) in \( t + 1 \) (this includes the case of another impression being matched to one of them). Since matching \( j_1 \) or \( j_2 \) in \( t \) are mutually exclusive, we have

\[
P(\text{match } i_2 \text{ with } j_1 \text{ or } j_2 \text{ in } t) \leq \frac{1}{n} \left[ \frac{|I_1|}{n} (1 - P(\text{match } j_1 \text{ or } j_2 \text{ in } [t + 2, n]))
\right.
\]

\[
+ (1 - P(\text{match some } i \in I_1 \text{ with } j_1 \text{ or } j_2 \text{ in } t + 1)) \right],
\]

which is equivalent to

\[
|I_1| \sum_{\tau = t+2}^{n} Z_{N,j}^{\tau} + nZ_{t+1,j}^{t+1} + n^2 Z_{t+2,j}^{t} \leq |I_1| + n.
\]

As with inequalities (8), if we attempt to naively extend this argument by considering a larger set \( I_2 \) instead of the single impression \( i_2 \), we simply get redundant inequalities. However, we can generalize (11) using the same argument for (10): For any \( I \subseteq N \) with \(|I| = r \leq n - 1 \) and any \( J \subseteq V \) with \(|J| = h \leq n - 1 \), we obtain the inequalities

\[
r \sum_{\tau = t+h}^{n} Z_{N,j}^{\tau} + nZ_{t,j}^{t} + \sum_{\tau = t}^{r+h-2} n^{t+h+\tau} Z_{t+r,j}^{t} \leq r + \sum_{\tau = 1}^{h-1} n^\tau,
\]

\[
\forall J \subseteq V, |J| = h \in [n-1], \ I \subseteq N, |I| = r \in [n-1], t \in [n-h], i_t, \ldots, i_{t+h-2} \in N.
\]

**Theorem 6.** Constraints (12) are facet-defining for \( Q \).

For a proof of this theorem, see the Appendix.

In inequalities (12), we do not consider \( I = N \). Suppose we apply the same argument for (11) in this case; we then obtain

\[
 n \sum_{\tau = t+h}^{n} Z_{N,j}^{\tau} + nZ_{N,j}^{t+h-1} + \sum_{\tau = t}^{r+h-2} n^{t+h+\tau} Z_{t+r,j}^{t} \leq n + \sum_{\tau = 1}^{h-1} n^\tau.
\]

Dividing by \( n \), we get

\[
\sum_{\tau = t+h}^{n} Z_{N,j}^{\tau} + Z_{N,j}^{t+h-1} + \sum_{\tau = t}^{r+h-2} n^{t+h+\tau} Z_{t+r,j}^{t} \leq 1 + \sum_{\tau = 1}^{h-1} n^{\tau-1},
\]
Theorem 7. Inequalities (13) identified by \((h, r, t, q)\) are facet-defining for \(Q\) when \(h \in [2, n-1], r \in [n-1], t \in [n-h] \) and \(q \in [0, h-2]\).

Finally, we show the following complexity result for this general family of facet-defining inequalities.
Proposition 8. It is NP-hard to separate inequalities (12), and thus also (13).

Proof. Fix $h = r$ and $t$. Suppose we have a solution $z$ that is zero (or constant) in all values except for stage $t + h - 1$. In this case, the separation problem for this $h$, $r$ and $t$ is equivalent to

$$\max \{ Z_{t+1}^{t+h-1} : I \subseteq N, J \subseteq V, |I| = |J| = h = r \}.$$  

This is a weighted version of the maximum balanced biclique problem, which is NP-hard [6]. For $h \neq r$, the problem can be transformed to make the two cardinalities equal.  

4 Computational Results

In this section we outline the experiments we conducted to test the bounds given by our relaxations.

4.1 Description of Experiments

Our main experimental goal is testing the effectiveness of our new time-indexed relaxations and comparing the new bounds given by these relaxations to several benchmarks. As a secondary goal, we also study a heuristic policy implied by our relaxation and compare it with an effective heuristic policy from the literature.

The best empirical bound previously known for OBM is the LP (4) with additional inequalities (5) [22]. Our results in the previous section establish that an LP in the space of $z_{ij}$ variables with inequalities (6) and (8) is guaranteed to be no worse:

$$\max_{z \geq 0} \left\{ \sum_{i \in N} \sum_{j \in V} \sum_{t \in [n]} w_{t,ij} z_{t,ij} : (6), (8) \right\}. \tag{14}$$

So we compare these two bounds to determine how much of an improvement the latter LP (14) offers over the former.

In addition, we would like to examine if some of the other inequalities we introduce can further improve the bound. However, testing these additional inequality classes involves computational challenges. In particular, the LP’s dimension grows as $n^3$, implying a relatively large number of variables even for moderately sized instances. This practically limits both the number of inequalities we consider, and the actual number we can dynamically add to the LP. To this end, we test adding inequalities (9) to (14); these inequalities are still polynomially many, $\Theta(n^5)$, and relatively efficient to separate over. We also considered including inequalities (17), that is, the special case of (10) with $h = n - 1$ and $t = 1$, as they are also simple to separate over despite numbering $\Theta(n^n)$. However, our preliminary experiments revealed numerical difficulties with these inequalities; the smallest non-zero coefficient is 1, while the largest is $n^n - 1$, and although these numbers (and all of the coefficients and right-hand sides of our inequalities) require $O(n \log n)$ space in binary representation and are thus of polynomial size, in practical terms these differences in scale make it difficult to even determine whether a particular inequality is violated, and thus to separate over the entire family. We therefore did not include these inequalities in our experiments.

As for lower bounds given by heuristic policies, the best performing policy tested in [22] uses dual multipliers from the LP (4) with additional inequalities (5) to create a time-dependent ranking policy. In a similar fashion, we can devise a policy from the LP (14). Denote by $\lambda_{t,ij} \geq 0$ and $\mu_{t,ij} \geq 0$ the dual multipliers corresponding to constraints (6) and (8) respectively. Along the lines of [22] and similar approximate DP approaches, we construct an approximation of the true value function (1): Interpret each $\lambda_{t,ij}$ as the value of having an impression of type $i$ appear in period $t$, and similarly interpret each $\mu_{t,ij}$ as the value of having impression $i$ appear in period
rubrics: do not accommodate weights that vary over time. We generate instances with the following static edges; the static weights are required because the benchmarks we use to compare against

All of the instances we tested have $n = m = T$, with binary edge weights constant over time, $w_{ij} = w_{ij} \in \{0, 1\}$. In other words, all the instances are max-cardinality OBM problems with static edges; the static weights are required because the benchmarks we use to compare against do not accommodate weights that vary over time. We generate instances with the following rubrics:

1. 20 small instances with $n = 10$, each one randomly generated by having a possible edge in $N \times V$ be present independently with a probability of 25%, so the expected average degree is 2.5.
2. 20 large, dense instances with $n = 50$, each one randomly generated by having a possible edge in $N \times V$ be present independently with a probability of 10%, so the expected average degree is 5.
3. 20 large, sparse instances with $n = 50$, each one randomly generated by having a possible edge in $N \times V$ be present independently with probability of 5%, so the expected average degree is 2.5.

4.2 Instance Design and Implementation

By imposing the constraints from (2) on this approximation of the value function, we obtain the dual of (14):

$$
\min_{v \geq 0} \ E_\eta[v_i(\eta, V)]
\quad \text{s.t.} \quad v_i(i, S \cup j) - E_\eta[v_{i-1}(\eta, S)] \geq w_{ij}^t, \quad (15)$$

$$
\min_{\lambda, \mu \geq 0} \ \sum_{t \in [n]} \left( E_\eta[\lambda_t^i] + \sum_{j \in V} E_\eta[\mu_{nj}^t] \right)
\quad \text{s.t.} \quad \lambda_t^i + \mu_{ij}^t + \sum_{\tau \in [t-1]} E_\eta[\mu_{nj}^\tau] \geq w_{ij}^t.
$$

Furthermore, by replacing (15) in the DP recursion (1) for a state $(t, i, S)$, we get the heuristic policy

$$
\arg \max \left\{ \max_{j \in S} \left( w_{ij}^t + E_\eta[v_{i-1}(\eta, S \setminus j)] \right), E_\eta[v_{i-1}(\eta, S)] \right\}
\approx \arg \max \left\{ \max_{j \in S} \left( w_{ij}^t + \sum_{\tau \in [t-1]} \left( E_\eta[\lambda_{t-\tau}^i] + \sum_{\ell \in S \setminus j} E_\eta[\mu_{n\ell}^{t-\tau}] \right) \right), \sum_{\tau \in [t-1]} E_\eta[\lambda_{t-\tau}^i] + \sum_{\ell \in S} \sum_{\tau \in [t-1]} E_\eta[\mu_{n\ell}^{t-\tau}] \right\}
= \arg \max \left\{ \max_{j \in S} \left( w_{ij}^t - \sum_{\tau \in [t-1]} E_\eta[\mu_{n\ell}^{t-\tau}] \right), 0 \right\}. \quad (16)
$$

Intuitively, this policy evaluates the net benefit of a potential match of impression $i$ to ad $j$ in period $t$ as the match’s weight minus the value we give up by losing ad $j$ in the subsequent remaining periods. The policy chooses the match with the largest such benefit (if positive), and otherwise discards the impression. Structurally, this heuristic policy is similar to dynamic bid price policies used in network revenue management models with independent demand; see e.g. [1, 16, 23] and references therein.

Finally, we include as additional benchmarks the optimal value given by the DP recursion (1) (for small instances where it can be computed), as well as the max-weight expected off-line matching, the expected value of the matching we would choose if we could observe the entire sequence of realized impressions before making a decision. This latter benchmark is also an upper bound on the optimal value, as it relaxes non-anticipativity.
For any experiment requiring simulation, including computing the expected value of the heuristic policies and the max-weight off-line matching, we used 20,000 simulations and report the sample mean and sample standard deviation.

For small instances, all the bound experiments took a few seconds on average. For the larger instances, we solved the benchmark LP’s following the approach from [22]. For the new bounds, the relatively large dimension of these LP’s coupled with the large number of inequalities (8) and (9) necessitated constraint generation. To solve (14), we separated over inequalities (8) using a straightforward greedy separation routine, and solved the LP’s usually in under a minute, 40 seconds on average. After solving this LP, we switched to separating over (9); however, as this family includes \( \Theta(n^5) \) inequalities (roughly \( 10^8 \) for \( n = 50 \)), we imposed both a limit of 100 inequalities added per value of \( t \in [n - 2] \) and a total time limit of 40 minutes.

### 4.3 Summary of Results

Table 1 summarizes the experiment results. For each instance class, in each row we present the geometric mean of each bound or policy’s ratio to a fixed benchmark – the DP value for small instances and the max-weight expected off-line matching for large ones. We also report the sample standard deviation of these ratios in parenthesis.

<table>
<thead>
<tr>
<th>Bound/Policy</th>
<th>Small</th>
<th>Large Dense</th>
<th>Large Sparse</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4) + (5)</td>
<td>1.0831 (0.0179)</td>
<td>1.0239 (0.0094)</td>
<td>1.0809 (0.0134)</td>
</tr>
<tr>
<td>(14)</td>
<td>1.0455 (0.0096)</td>
<td>0.9776 (0.0068)</td>
<td>1.0287 (0.0072)</td>
</tr>
<tr>
<td>(14) + (9)</td>
<td>1.0423 (0.0080)</td>
<td>0.9770 (0.0073)</td>
<td>1.0278 (0.0070)</td>
</tr>
<tr>
<td>Off-Line Exp. Matching</td>
<td>1.0205 (0.0088)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(1)</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Policy (16)</td>
<td>0.9962 (0.0023)</td>
<td>0.9312 (0.0050)</td>
<td>0.9653 (0.0078)</td>
</tr>
<tr>
<td>TD Ranking Policy [22]</td>
<td>0.9732 (0.032)</td>
<td>0.8785 (0.0110)</td>
<td>0.9198 (0.0109)</td>
</tr>
</tbody>
</table>

Table 1: Summary of experiment results.

We know from our results in the previous section that the bound given by (14) is guaranteed to outperform the bound given by (4) with (5). However, our results show that the improvement is significant, with the new bound cutting the gap by about 4% on average for small instances and approximately 5% for large ones. Furthermore, the improvement is consistent across all the tested instances; in particular, the two bounds never match.

The results for large, dense instances are particularly noteworthy; the new bound from (14) also beats the max-weight expected off-line matching, not only on average but in every instance. Our intuition for this result is the following. In dense instances, there is likely a perfect or near-perfect matching in every realization, and thus the off-line matching will be very close to \( n \) in expectation. Of course, even in a dense instance it may be that no online policy can guarantee a perfect or near-perfect matching, and explicitly accounting for temporal aspects of the problem, particularly as inequalities (8) do, captures this phenomenon and tightens the bound, unlike the off-line matching or the more static approach of the benchmark LP.

Interestingly, our results also reveal that the bound from (14) is not improved much with the addition of inequalities (9), especially considering the significant additional computing time. In light of these results, we also performed experiments to test the bound given by (6) and (9) only (without inequalities (8)). However, the resulting bounds were much looser, confirming that inequalities (8) are crucial to providing a tight bound.

In terms of policies, our new heuristic (16) clearly outperforms the time-dependent ranking policy, the best performing policy from [22]. This improvement occurs in every tested instance, a strong indication of the heuristic’s quality. The new policy is near-optimal for small instances, and significantly cuts the gap for large instances, by about 4% to 5% on average. This improvement in policy quality mirrors results in other areas, such as revenue management, where
heuristic policies derived from time-indexed relaxations also outperform policies stemming from “static” LP’s; see e.g. [1, 24].

5 Conclusions

This work presented a polyhedral study of the i.i.d. OBM problem. While several past results have used different LP relaxations, ours is the first to explicitly consider the time dimension. Among various benefits of the approach, this allows for the model to accommodate time-varying edge weights, and also allows us to elide the instance’s structure in the analysis, by capturing all of this information in the problem’s objective. Our study centers on the polyhedron of time-indexed achievable probabilities $Q$, and includes a large class of facet-defining inequalities for this polytope based on choosing complete bipartite subgraphs. Furthermore, our experiments confirm that the time-indexed approach offers significant benefits; the bound given by the simplest members of our proposed inequality family already significantly outperforms the best empirical bounds given by static LP’s, and a heuristic policy derived from this new bound also significantly outperforms the best policy based on a static relaxation.

Our results motivate a variety of questions for future work. For example, we would like to understand the structure of valid inequalities that are not based on complete bipartite subgraphs, to potentially further improve the dual bound. Using Fourier-Motzkin elimination and the software PORTA, we have derived the full description of $Q$ for small cases, such as $n = m = T = 3$. We observed many different inequalities, including some that are somewhat similar to our general family (7), so there may be a more general class to propose that still lends itself to analysis similar to ours.

Much of the literature on OBM studies the worst-case performance of heuristic policies based on relaxations. Although that was not our goal in this work, the positive empirical results we observed when implementing our new heuristic policy suggest a similar analysis for that policy, especially since it appears to differ in structural terms from many OBM heuristics. More generally, an interesting question is whether a polyhedral analysis similar to ours can be applied to derive new bounds and policies in related online matching and resource allocation contexts.

References


6 Appendix

6.1 Remaining Proofs

Proof of Theorem 5. The case $h = 1$ is already covered by the proof of Proposition 3. Consider the case $h = n - 1$ and $t = 1$; the other cases follow a similar construction of linearly independent points. Let $J = \{0, \ldots, n-2\}$, and assume without loss of generality that $i_\tau = i$ for all $\tau \in [n-1]$. The specific inequality is

$$Z_{n,J}^n + \sum_{\tau=1}^{n-1} n^{n-\tau} Z_{i,J}^\tau \leq 1 + \sum_{\tau=1}^{n-2} n^{\tau}. \quad (17)$$

We know that $z \in [0,1]^n$, but for the description of the points (and the proof) we will just consider the coordinates involved in the inequality, i.e., $z \in [0,1]^p$, where $p := (2n - 1)(n - 1)$. For the rest of the points, the construction is similar to the one in the proof of Proposition 3. Recall that $e_{k,j}^\tau$ denotes the canonical vector in $[0,1]^p$, i.e. a vector with a 1 in coordinate $(k,j,\tau)$ and zero elsewhere, indicating a match of impression $k$ with ad $j$ in stage $\tau$. Consider the elements of $J$ as an $(n-1)$-tuple, i.e., $(0, \ldots, n-2)$. For $j \in J$, we define

$$j + (0, \ldots, n-2) := (j, \ldots, j + n-2) \mod (n-1).$$

Any addition or substraction with $j \in J$ is modulo $(n-1)$ for the remainder of the proof. We denote the circulation of $J$ as the following set of $(n-1)$-tuples:

$$\text{circ}(J) := \{j + (0, \ldots, n-2)\}_{j \in J}$$
$$= \{(0, \ldots, n-2), (1, \ldots, n-2, 0), \ldots, (n-2, 0, \ldots, n-3)\}.$$

Note that circ$(J)$ can be viewed as a matrix. Each element of circ$(J)$ corresponds to a sequence of ads in the process from stage $n$ to stage $1$. Since we have $n$ stages and any of those sequences has size $n-1$, then clearly there is no matching in some stage or an element repeats. We now describe the family of linearly independent points.

I. Fix $k \in N$ and $j \in J$. In stage $n$, if node $k$ appears, then match it to node $j$, with probability $1/n$. For the remaining stages match according to $(j, j+1, \ldots, j+n-2) \in \text{circ}(J)$. In terms of probability, if $i$ appears in stage $n-1$, then it is matched to $j$ with probability $(1-1/n) \cdot 1/n$. For the rest, the probability is $1/n$. So, we have the point

$$\frac{1}{n} e_{k,j}^n + \frac{1}{n} (1 - 1/n) e_{i,j}^{n-1} + \frac{1}{n} \sum_{\tau=1}^{n-2} e_{i,j+\tau}^{n-\tau}. \quad (18)$$

By a simple calculation, it is easy to see that each of these points achieves the righthand side of (17). Since we chose an arbitrary $k \in N$ and $j \in J$, we have $n(n-1)$ points in this family.

II. Fix $j \in J$. In this family we repeat $j$ in stages $n-1$ and $n-2$. If $i$ appears in stage $n-1$, match it to node $j$ with probability $1/n$. If $i$ appears in stage $n-2$ and it did not appear in $n-1$, match it to $j$ with probability $(1-1/n) \cdot 1/n$. For the remaining stages match according to $(j+n-2, j+1, \ldots, j+n-3) \in \text{circ}(J)$; in particular, in stage $n$ match any $k \in N$ that appears with node $j+n-2$, in stage $n-3$ match $i$ to $j+1$ if it appears, and so forth. So, we have the point

$$\frac{1}{n} \sum_{k \in N} e_{k,j+n-2}^n + \frac{1}{n} e_{i,j}^{n-1} + \frac{1}{n} (1 - 1/n) e_{i,j}^{n-2} + \frac{1}{n} \sum_{\tau=1}^{n-3} e_{i,j+\tau}^{n-\tau}. \quad (19)$$

By a simple calculation, we get the right-hand side of (17). Since we chose an arbitrary $j \in J$, we have $n-1$ points in this family.
III. Fix $j \in J$; in this family we have two different options in stage $n - 3$. If $i$ appears in stage $n - 1$, match it to $j$ with probability $1/n$. If $i$ appears in stage $n - 2$, match it to $j + 1$, also with probability $1/n$. If $i$ appears in stage $n - 3$, match it to $j + 1$ with probability $(1 - 1/n) \cdot 1/n$, or if $j + 1$ was matched in stage $n - 2$, then to node $j$ with probability $(1 - 1/n) \cdot 1/n^2$. For the remaining stages match according to $(j + n - 2, j, j + 1, \ldots, j + n - 3)$; in stage $n$ match any $k \in N$ that appears to $j + n - 2$, in stage $n - 4$ match $i$ to $j + 2$ if it appears, and so forth. So, we have the point

$$\frac{1}{n} \sum_{k \in N} e_{k,j+n-2}^n + \frac{1}{n} e_{i,j}^{n-1} + \frac{1}{n} e_{i,j+1}^{n-2} \cdot \left(1 - \frac{1}{n}\right) \left[\frac{1}{n^2} e_{i,j}^{n-3} + \frac{1}{n} e_{i,j+1}^{n-3}\right] + \frac{1}{n} \sum_{\tau=1}^{n-4} e_{i,j+\tau+1}^{n-n-\tau-3} \quad (20)$$

By a simple calculation, we get the right-hand side of (17). Since we chose an arbitrary $j \in J$, we have $n - 1$ points in this family.

IV. Fix $j \in J$ and stage $s \in [n - 4]$; the previous family can be generalized for stage $s$, but increasing the number of options, i.e., in stage $s$ we have $n - s - 1$ options from the previous stages. If $i$ appears in stage $n - 1$, match it to $j$ with probability $1/n$, if $i$ appears in stage $n - 2$, match it to $j + 1$ with probability $1/n$, and continue in this way until stage $s + 1$, where if $i$ appears, match it to node $j + n - s - 2$ with probability $1/n$. If $i$ appears in stage $s$, we consider ads $(j + n - s - 2, \ldots, j + 1, j)$ in this order of priority, so that $i$ is matched to $j + n - s - 2$ with probability $(1 - 1/n) \cdot 1/n$; each subsequent ad's probability of being matched to $i$ decreases exponentially until $j$, which has probability $(1 - 1/n) \cdot 1/n^{s-1}$. For the remaining stages (including stage $n$) match according to $(j + n - 2, j, j + 1, \ldots, j + n - 3)$; in stage $n$ match any $k \in N$ that appears with $j + n - 2$, in $s - 1$ match $i$ to $j + n - s - 1$ if it appears, etc. So, we have the point

$$\frac{1}{n} \sum_{k \in N} e_{k,j+n-2}^n + \frac{1}{n} \sum_{\tau=0}^{n-s-2} e_{i,j+\tau}^{n-\tau-1} \cdot \left(1 - \frac{1}{n}\right) \left[\sum_{\tau=0}^{n-s-2} \frac{1}{n^{\tau-s-1}} e_{i,j+\tau}^s\right] + \frac{1}{n} \sum_{\tau=1}^{s-1} e_{i,j+n-s-2+\tau}^{n-s-\tau} \quad (21)$$

The left-hand side of (17) evaluated at this point is

$$1 + \sum_{\tau=s+1}^{n-1} \frac{n^{n-\tau}}{n^\tau} + \left(1 - \frac{1}{n}\right) \sum_{\tau=0}^{n-s-2} \frac{n^{n-s}}{n^{\tau-s-1}} + \sum_{\tau=1}^{s-1} \frac{n^{n-\tau}}{n^\tau} = 1 + \sum_{\tau=1}^{n-2} n^\tau.$$

Finally, since we chose an arbitrary $j \in J$ and $s \in [n - 4]$, we have $(n - 1)(n - 4)$ points in this family.

V. Fix $j \in J$. For this family we do not match in stage $n$, and in the remaining stages we match according to $(j, j + 1, \ldots, j + n - 2) \in \text{circ}(J)$. If $i$ appears in stage $n - 1$ match it to $j$ with probability $1/n$, if $i$ appears in stage $n - 2$ match it to $j + 1$, and so on. So we have the point

$$\frac{1}{n} \sum_{\tau=0}^{n-2} e_{i,j+\tau}^{n-\tau-1} \quad (22)$$

By a simple calculation, we get the right-hand side of (17). Finally, since we chose an arbitrary $j \in J$, then we have $n - 1$ points in this family.

With these families, we have $p$ points in total. Denote by $(k, j, \tau)$ the index of a vector $z \in [0, 1]^p$, which indicates that $k \in N$ is matched to $j \in J$ in stage $\tau$. In any of these points consider the following order of components (starting from the first one): $(1, 0, n)$, $(1, 1, n)$, $\ldots$, $(1, n - 2, n)$, $\ldots$, $(n, 0, n)$, $\ldots$, $(n, n - 2, n)$, $(i, 0, n - 1)$, $\ldots$, $(i, n - 2, n - 1)$, $\ldots$, $(i, 0, 1)$, $\ldots$, $(i, n - 2, 1)$. 17
The rest of the proof consists of showing that these families define a set of linearly independent points, and we prove this using Gaussian elimination. Arrange these points as column vectors in a matrix $A$,

$$A = [\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}, \mathbf{V}] = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where $B_1$ is a $n(n-1) \times n(n-1)$ diagonal matrix with entries $1/n$. These columns can be used to make $B_2$ a zero matrix, yielding

$$\bar{A} = \begin{pmatrix} B_1 & 0 \\ B_3 & C \end{pmatrix}.$$

Consider how the columns from families II, III, and IV look like after this elimination procedure (family V is not affected). Fix $g \in J$ and sum every point (18) over $k \in N$; this yields

$$\frac{1}{n} \sum_{k \in N} e_{k,g}^n + \left(1 - \frac{1}{n}\right) e_{i,g}^{n-1} + \frac{1}{n} \sum_{\tau=1}^{n-2} e_{i,g+\tau}^{n-1}. \quad (23)$$

II$^a$. Pick the point (19) associated with $g + 1 \in J$,

$$\frac{1}{n} \sum_{k \in N} e_{k,g}^n + \frac{1}{n} e_{i,g+1} + \frac{1}{n} \left(1 - \frac{1}{n}\right) e_{i,g+1}^{n-2} + \frac{1}{n} \sum_{\tau=1}^{n-3} e_{i,g+1+\tau}^{n-2} + \frac{1}{n} \left(1 - \frac{1}{n}\right) e_{i,g}^{n-1} - \frac{1}{n} \sum_{\tau=1}^{n-2} e_{i,g+\tau}^{n-1} \quad (24)$$

Subtract (23) from (24) to get

$$\frac{1}{n} e_{i,g+1}^{n-1} + \left(1 - \frac{1}{n}\right) e_{i,g}^{n-1} + \frac{1}{n} \left(1 - \frac{1}{n}\right) e_{i,g+1}^{n-2} + \frac{1}{n} \sum_{\tau=2}^{n-3} e_{i,g+\tau}^{n-2} - \frac{1}{n} \sum_{\tau=1}^{n-2} e_{i,g+\tau}^{n-1} \quad (25)$$

III$^a$. Pick the point (20) associated with $g + 1 \in J$,

$$\frac{1}{n} \sum_{k \in N} e_{k,g}^n + \frac{1}{n} e_{i,g+1} + \frac{1}{n} e_{i,g+2} + \left(1 - \frac{1}{n}\right) \left[\frac{1}{n^2} e_{i,g+1}^{n-3} + \frac{1}{n} e_{i,g+2}^{n-3}\right] + \frac{1}{n} \sum_{\tau=1}^{n-4} e_{i,g+\tau+2}^{n-3}. \quad (26)$$

Subtract (23) from (26) to get

$$\left(-1 + \frac{1}{n}\right) e_{i,g}^{n-1} + \frac{1}{n} e_{i,g+1}^{n-1} + \frac{1}{n} e_{i,g+2}^{n-1} - e_{i,g+1}^{n-2} + \left(1 - \frac{1}{n}\right) \left[\frac{1}{n^2} e_{i,g+1}^{n-3} + \frac{1}{n} e_{i,g+2}^{n-3}\right] - \frac{1}{n} \sum_{\tau=3}^{n-3} e_{i,g+\tau}^{n-1} \quad (27)$$

IV$^a$. Pick the point (21) associated with $g + 1 \in J$ and any $s \in [n-4],$

$$\frac{1}{n} \sum_{k \in N} e_{k,g}^n + \sum_{\tau=0}^{n-s-2} \frac{1}{n} e_{i,g+\tau+1}^{n-\tau-1} + \left(1 - \frac{1}{n}\right) \left[\sum_{\tau=0}^{n-s-2} \frac{1}{n} e_{i,g+\tau+1}^{n-s-\tau-1}\right] + \frac{1}{n} \sum_{\tau=1}^{n-s-1} e_{i,g+n-1-s+\tau}^{n-s-\tau}. \quad (28)$$
Subtract (23) from (28) to get

\[
\sum_{\tau=0}^{n-s-2} \frac{1}{n} e_{i,g+\tau+1}^{n-s-1} + \left(1 - \frac{1}{n}\right) \left[\sum_{\tau=0}^{n-s-2} \frac{1}{n^{n-s-\tau-1}} e_{i,g+\tau+1}^s\right]
\]

\[
+ \frac{1}{n} \sum_{\tau=1}^{s-1} e_{i,g+n-1-s+\tau}^s - \left(1 - \frac{1}{n}\right) e_{i,g}^{n-1} - \sum_{\tau=1}^{n-2} e_{i,g+\tau}^{n-1},
\]

which is equivalent to

\[
\left(-1 + \frac{1}{n}\right) e_{i,g}^{n-1} + \frac{1}{n} e_{i,g+1}^{n-1} + \frac{1}{n} e_{i,g+2}^{n-2} - e_{i,g+1}^{n-2} + \left(1 - \frac{1}{n}\right) \left[\frac{1}{n^2} e_{i,g+1}^{n-3} - \frac{1}{n^2} e_{i,g+2}^{n-3}\right]
\]

\[
+ \left(-1 + \frac{1}{n}\right) e_{i,g+1}^{n-2} - e_{i,g+2}^{n-2} + \left(-1 + \frac{1}{n}\right) e_{i,g+1}^{n-1} + \left(\frac{1}{n} - \frac{1}{n^2} - 1\right) e_{i,g+1}^{n-2}
\]

\[
- \left(-1 + \frac{1}{n}\right) \sum_{\tau=2}^{n-2} e_{i,g+\tau}^{n-1},
\]

which is equivalent to

\[
\left(-1 + \frac{1}{n}\right) e_{i,g+1}^{n-2} + \frac{1}{n} e_{i,g+2}^{n-2} + \left(\frac{1}{n^2} - \frac{1}{n^3}\right) e_{i,g+1}^{n-3} - \frac{1}{n^2} e_{i,g+2}^{n-3}.
\]

IVb. For every \(s \in [n - 4]\), subtract (25) from (29),

\[
\left(-1 + \frac{1}{n}\right) e_{i,g}^{n-1} + \frac{1}{n} e_{i,g+1}^{n-1} + \sum_{\tau=1}^{n-s-2} \left[\frac{1}{n} e_{i,g+\tau+1}^{n-s-1} - e_{i,g+\tau}^{n-s-1}\right]
\]

\[
+ \left(1 - \frac{1}{n}\right) \left[\sum_{\tau=0}^{n-s-2} \frac{1}{n^{n-s-\tau-1}} e_{i,g+\tau+1}^s\right] - e_{i,g+n-s-1}^s
\]

\[
+ \left(-1 + \frac{1}{n}\right) \sum_{\tau=2}^{n-2} e_{i,g+\tau}^{n-1},
\]

\[
- \frac{1}{n} e_{i,g+1}^{n-1} - \left(-1 + \frac{1}{n}\right) e_{i,g}^{n-1} - \left(\frac{1}{n} - \frac{1}{n^2} - 1\right) e_{i,g+1}^{n-2}
\]

\[
- \left(-1 + \frac{1}{n}\right) \sum_{\tau=2}^{n-2} e_{i,g+\tau}^{n-1},
\]
which is equivalent to
\[
\left( -\frac{1}{n} + \frac{1}{n^2} \right) e_{i,g+1}^{n-2} + \frac{1}{n} e_{i,g+2}^{n-2} + \frac{1}{n} \sum_{\tau=2}^{n-s-2} \left[ e_{i,g+\tau+1}^{n-\tau-1} - e_{i,g+\tau}^{n-\tau-1} \right] \\
+ \left( 1 - \frac{1}{n} \right) \left[ \sum_{\tau=0}^{n-s-3} \frac{1}{n^{n-\tau-s-1}} e_{i,g+\tau+1}^{*} \right] - \frac{1}{n^2} e_{i,g+n-s-1}^{*}. \tag{31}
\]

IVc. For every \( s \in [n-4] \), subtract (30) from (31),
\[
\left( -\frac{1}{n} + \frac{1}{n^2} \right) e_{i,g+1}^{n-2} + \frac{1}{n} e_{i,g+2}^{n-2} + \frac{1}{n} \sum_{\tau=2}^{n-s-2} \left[ e_{i,g+\tau+1}^{n-\tau-1} - e_{i,g+\tau}^{n-\tau-1} \right] \\
+ \left( 1 - \frac{1}{n} \right) \left[ \sum_{\tau=0}^{n-s-3} \frac{1}{n^{n-\tau-s-1}} e_{i,g+\tau+1}^{*} \right] - \frac{1}{n^2} e_{i,g+n-s-1}^{*}
\]
which is equivalent to
\[
\frac{1}{n} e_{i,g+3}^{n-3} - \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+1}^{n-3} + \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+2}^{n-3} + \frac{1}{n} \sum_{\tau=3}^{n-s-2} \left[ e_{i,g+\tau+1}^{n-\tau-1} - e_{i,g+\tau}^{n-\tau-1} \right] \\
+ \left( 1 - \frac{1}{n} \right) \left[ \sum_{\tau=0}^{n-s-3} \frac{1}{n^{n-\tau-s-1}} e_{i,g+\tau+1}^{*} \right] - \frac{1}{n^2} e_{i,g+n-s-1}^{*}. \tag{32}
\]

IVd. For every \( s \in [n-5] \), subtract (32) corresponding to \( s+1 \) from (32) corresponding to \( s \),
\[
\frac{1}{n} e_{i,g+3}^{n-3} - \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+1}^{n-3} + \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+2}^{n-3} + \frac{1}{n} \sum_{\tau=3}^{n-s-2} \left[ e_{i,g+\tau+1}^{n-\tau-1} - e_{i,g+\tau}^{n-\tau-1} \right] \\
+ \left( 1 - \frac{1}{n} \right) \left[ \sum_{\tau=0}^{n-s-3} \frac{1}{n^{n-\tau-s-1}} e_{i,g+\tau+1}^{*} \right] - \frac{1}{n^2} e_{i,g+n-s-1}^{*}
\]
which is equivalent to
\[
\frac{1}{n} e_{i,g+n-s-1}^{s+1} + \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+n-s-2}^{s+1} + \left( 1 + \frac{1}{n} \right) \left[ \sum_{\tau=0}^{n-s-4} \frac{1}{n^{n-\tau-s-2}} e_{i,g+\tau+1}^{s+1} \right] \\
- \frac{1}{n^2} e_{i,g+n-s-1}^{s} + \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+n-s-2}^{s} + \left( 1 - \frac{1}{n} \right) \left[ \sum_{\tau=0}^{n-s-4} \frac{1}{n^{n-\tau-s-1}} e_{i,g+\tau+1}^{s} \right]. \tag{33}
\]

For \( s = n-4 \), we do not need this step, since from (32) we have
\[
\frac{1}{n} e_{i,g+3}^{n-3} + \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+2}^{n-3} + \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+1}^{n-3} \\
- \frac{1}{n^2} e_{i,g+3}^{n-4} + \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+2}^{n-4} + \left( \frac{1}{n^2} - \frac{1}{n^3} \right) e_{i,g+1}^{n-4}.
\]
Observe that for any \( s \in \{n - 4\} \) and \( g \in J \), we can multiply row \((i, g, s + 1)\) by \(-1/n\) and we get the entry in row \((i, g, s)\).

II\(^b\). Finally, pick a point \((22)\) in family \(V\) for \( g \in J\),

\[
\frac{1}{n} e_{i,g}^{n-1} + \frac{1}{n^2} e_{i,g+1}^{n-2} + \ldots + \frac{1}{n^{n-2}} e_{i,g+n-2}.
\] (34)

Now multiply (34) by \((1 - n)\) and subtract it from (25) for \( g \in J\), yielding

\[
\frac{1}{n} e_{i,g}^{n-1} + \left(-1 + \frac{1}{n}\right) e_{i,g+1}^{n-1} + \left(\frac{1}{n} - \frac{1}{n^2} - 1\right) e_{i,g+2}^{n-2} + \ldots + \left(-1 + \frac{1}{n}\right) \sum_{\tau=2}^{n-2} e_{i,g+\tau}^{n-1} - \frac{1}{n} \sum_{\tau=0}^{n-2} e_{i,g+\tau}^{n-1},
\]

which is equivalent to

\[
\frac{1}{n} e_{i,g+1}^{n-1} - \frac{1}{n^2} e_{i,g+1}^{n-2}.
\]

Since we have \( g + 1 \) in those two stages, we have a general expression for any \( g \in J\),

\[
\frac{1}{n} e_{i,g}^{n-1} - \frac{1}{n^2} e_{i,g}^{n-2}.
\] (35)

As before, we can multiply row \((i, g, n - 1)\) by \(-1/n\) to get the entry in row \((i, g, n - 2)\).

Now, we can organize the points in \(C\) as

\[
C = [V, II^b, III^b, IV_{n-4}^d, \ldots, IV_s^d, \ldots, IV_1^d],
\]

where \(IV_s^d\) corresponds to the block of points \((g \in J)\) with \(s \in \{n - 4\}\). \(C\) has the form

\[
C = \begin{pmatrix}
C_{n-1} & D_{n-2} & 0 & 0 & 0 & \ldots & 0 & 0 \\
C_{n-2} & -\frac{1}{n} D_{n-2} & D_{n-3} & 0 & 0 & \ldots & 0 & 0 \\
C_{n-3} & 0 & -\frac{1}{n} D_{n-3} & D_{n-4} & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
C_2 & 0 & 0 & 0 & 0 & \ldots & -\frac{1}{n} D_2 & D_1 \\
C_1 & 0 & 0 & 0 & 0 & \ldots & 0 & -\frac{1}{n} D_1
\end{pmatrix},
\]

where every \(C_i\) and \(D_i\) are \textit{circulant} matrices \([15]\) of size \((n - 1) \times (n - 1)\). Since the determinant is invariant under elementary row and column operations, we can perform Gaussian elimination (of rows) from bottom to top, and we get

\[
\tilde{C} = \begin{pmatrix}
\tilde{C}_{n-1} & 0 & 0 & 0 & \ldots & 0 & 0 \\
\tilde{C}_{n-2} & -\frac{1}{n} D_{n-2} & 0 & 0 & \ldots & 0 & 0 \\
\tilde{C}_{n-3} & 0 & -\frac{1}{n} D_{n-3} & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\tilde{C}_2 & 0 & 0 & 0 & 0 & \ldots & -\frac{1}{n} D_2 & 0 \\
\tilde{C}_1 & 0 & 0 & 0 & 0 & \ldots & 0 & -\frac{1}{n} D_1
\end{pmatrix},
\]
where
\[
\tilde{C}_1 = C_1 + n(C_2 - \cdots n(C_{n-2} + nC_{n-1})\cdots) = \text{circ}(1/n, 1, n^2, \ldots, n^{n-3}),
\]
\[D_{n-2} = \text{circ}(1/n, 0, \ldots, 0),
\]
\[D_{n-3} = \text{circ}\left(\frac{1}{n}, -\frac{1}{n} + \frac{1}{n^2}, 0, \ldots 0\right)
\]
\[D_s = \text{circ}\left(\frac{1}{n}, -\frac{1}{n} + \frac{1}{n^2}, -\frac{1}{n^2} + \frac{1}{n^3}, \ldots, -\frac{1}{n^{n-s-2}} + \frac{1}{n^{n-s-1}}, 0, \ldots 0\right)
\]
\[D_1 = \text{circ}\left(\frac{1}{n}, -\frac{1}{n} + \frac{1}{n^2}, -\frac{1}{n^2} + \frac{1}{n^3}, \ldots, -\frac{1}{n^{n-3}} + \frac{1}{n^{n-2}}\right)
\]
For \(\tilde{C}_1\), the last entry, \(n^{n-3}\), is greater than the sum of the remaining entries. The same applies for \(D_s\), with entry \(1/n\). Due to Proposition 18 in [15] all these matrices are nonsingular, so \(C\) is nonsingular, and therefore \(C\) is nonsingular. This implies that \(A\) is nonsingular, showing that these points are linearly independent, and proceeding in the same way as we did in the proof of Proposition 3 for the remaining components, we can show (17) is facet-defining.

Proof of Theorem 6. The proof is similar to Theorem 5. Again, assume \(h = n - 1\) and \(t = 1\); the remaining cases follow a similar construction of linearly independent points. Without loss of generality we can assume that \(i_\tau = i\) for all \(\tau \in [n-2]\). So, we have an inequality of the form
\[rZ_{N,j}^n + nZ_{i,j}^{n-1} + \sum_{\tau = 1}^{n-2} n^{n-\tau}Z_{i,j}^{1,\tau} \leq r + \sum_{\tau = 1}^{n-2} n^{1,\tau}.
\]  (36)

We construct the following linearly independent points.

I. Fix \(k \in N\) and \(j \in J\). In stage \(n\), if node \(k\) appears, match it to node \(j\), with probability \(1/n\). For the remaining stages match according to \((j, j+1, \ldots, j+n-2) \in \text{circ}(J)\). In terms of probability, if any \(k' \in I\) appears in stage \(n-1\), then it is matched to \(j\) with probability \((1 - 1/n) \cdot 1/n\), so the probability of matching \(j\) in stage \(n-1\) is \((1 - 1/n) \cdot r/n\). For the rest, the probability is \(1/n\). So, we have the point
\[
\frac{1}{n} \epsilon_{k,j}^n + \frac{1}{n} \left(1 - \frac{1}{n}\right) \sum_{k' \in I} \epsilon_{k',j}^{n-1} + \frac{1}{n} \sum_{\tau = 1}^{n-2} \epsilon_{i,j+\tau}^{n-1,\tau}.
\]  (37)

By a simple calculation, it is easy to see that each of these points achieves the right-hand side of (36). Since we chose any arbitrary \(k \in N\) and \(j \in J\), we have \(n(n-1)\) points in this family.

II. Fix \(j \in J\) and \(k \in I\). In this family we repeat the same ad to match in stages \(n-1\) and \(n-2\). If \(k\) appears in stage \(n-1\), match it to \(j\) with probability \(1/n\). Then, if \(i\) appears in stage \(n-2\) and \(k\) did not appear in \(n-1\), match it to \(j\) with probability \((1 - 1/n) \cdot 1/n\). For the remaining stages match according to \((j + n-2, j+1, \ldots, j+n-3) \in \text{circ}(J)\); in stage \(n\) match any \(k' \in N\) that appears with \(j + n-2\), in stage \(n-3\) match \(i\) to \(j + 1\) if it appears, and so on. So, we have the point
\[
\frac{1}{n} \sum_{k' \in N} \epsilon_{k',j+n-2}^n + \frac{1}{n} \epsilon_{k,j}^{n-1} + \frac{1}{n} \left(1 - \frac{1}{n}\right) \epsilon_{i,j}^{n-2} + \frac{1}{n} \sum_{\tau = 1}^{n-3} \epsilon_{i,j+\tau}^{n-2,\tau}.
\]  (38)

By a simple calculation, we get the right-hand side of (17). Finally, since we chose an arbitrary \(j \in J\) and \(k' \in I\), we have \(r(n-1)\) points in this family.
V. Fix \( j \in J \). In this family we do not match in stage \( n \), and in the remaining stages we match according to a vector in \( \text{circ}(J) \). If any \( k \in I \) appears in stage \( n-1 \), match it to \( j \) with probability \( \frac{1}{n} \), if \( i \) appears in stage \( n-2 \), match it to node \( j+1 \), and so forth. So we have the point

\[
\frac{1}{n} \sum_{k \in I} e_{k,j}^{n-1} + \frac{1}{n} \sum_{\tau=1}^{n-2} e_{i,j+\tau}^{n-\tau-1}.
\]  

By a simple calculation, we get the right-hand side of (17). Since we chose an arbitrary \( j \in J \), we have \( n-1 \) points in this family.

Families III, and IV remain the same as in the proof of Theorem 5, so in total we have \( (n-1)(r+2n-2) \) points. The rest of the proof follows the same argument as Theorem 5.

**Proof of Theorem 7.** The proof follows the same argument as the previous two theorems, the only difference being that the collection of points given by policies corresponding to \( I \) is bigger; however, they still form a diagonal block, so we can apply the same procedure.

### 6.2 Detailed Experiment Results

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<th>Instance</th>
<th>(4) + (5)</th>
<th>(14)</th>
<th>(14) + (9)</th>
<th>Exp. Matching</th>
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<th>(16)</th>
<th>TDR Policy [22]</th>
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Table 2: Experiment results for small instances.
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<th>(14)</th>
<th>(14) + (9)</th>
<th>Exp. Matching</th>
<th>(16)</th>
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Table 3: Experiment results for large, dense instances.

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Table 4: Experiment results for large, sparse instances.