Dynamic LP Games with Risk-Averse Players

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joint with Nelson Uhan (USNA)
Sharing and Risk Aversion Don’t Mix: Dynamic LP Games with Risk-Averse Players

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Motivation

California cut flower industry (Nguyen/Toriello/Dessouky/Moore 13)

- Study of potential consolidation center estimated up to 1/3 reduction in transport costs. Issue deemed existential by some in industry.
  - Results presented before U.S. Congress.
  - Application submitted to USDOT for discretionary grant.
Motivation
California cut flower industry (Nguyen/Toriello/Dessouky/Moore 13)

▶ Study of potential consolidation center estimated up to 1/3 reduction in transport costs. Issue deemed existential by some in industry.
  ▶ Results presented before U.S. Congress.
  ▶ Application submitted to USDOT for discretionary grant.

▶ However, many farmers skeptical of consolidation and wary of potential risks.

▶ **Looming question**: Can many disparate agents agree on cooperation and cost sharing? How does aversion to risk affect the ability to cooperate?
Outline

Introduction
   A Crash Course in Cooperative Games
   Primer on Coherent Risk Measures

Model Formulation
   The Risk-Averse Strong Sequential Core
   Implications on Cooperation
   Application: Risk-Averse Newsvendor Games

Parting Thoughts
Cost Allocation in OR Models
(Static) cooperative games and the core

- Players: \( N = \{1, \ldots, n\} \), e.g. retailers or producers.
- Cost function: \( f : 2^N \rightarrow \mathbb{R} \), e.g. joint venture cost when some subset of players cooperates.
Cost Allocation in OR Models
(Static) cooperative games and the core

- Players: $N = \{1, \ldots, n\}$, e.g. retailers or producers.

- Cost function: $f : 2^N \rightarrow \mathbb{R}$, e.g. joint venture cost when some subset of players cooperates.

- Goal: Assuming all players in $N$ cooperate, find a cost allocation $\chi \in \mathbb{R}^N$ that splits cost “fairly.”

- Usually models situations in which agents enter into binding agreement.
  - Especially when cooperation occurs over time.
Cost Allocation in OR Models
Cooperative games and the core

- **Core** (Gillies 59): Set of allocations $\chi$ that are
  
  efficient: $\sum_N \chi_i \geq f(N)$,

  stable: for $U \subseteq N$, no allocation $\xi$ has

  $\sum_U \xi_i \geq f(U)$ and $\xi_i < \chi_i$, $i \in U$.

  No player or coalition can do better by *defecting*. 
Cost Allocation in OR Models
Cooperative games and the core

- Core (Gillies 59): Set of allocations $\chi$ that are
  - efficient: $\sum_N \chi_i \geq f(N)$,
  - stable: for $U \subseteq N$, no allocation $\xi$ has
    $$\sum_U \xi_i \geq f(U) \quad \text{and} \quad \xi_i < \chi_i, \quad i \in U.$$  

  No player or coalition can do better by defecting.

- Equivalently described by
  $$\{\chi \in \mathbb{R}^N : \sum_N \chi_i \geq f(N); \quad \sum_U \chi_i \leq f(U), U \subseteq N\}.$$ 

- Non-empty core suggests cooperation is possible.
Some Application Examples
Production, inventory and supply chain management

- **Newsvendor and inventory centralization.**
  - Chen (09), Chen/Zhang (09), Hartman/Dror (96,05), Hartman/Dror/Shaked (00), Montrucchio/Scarsini (07), Müller/Scarsini/Shaked (02), Özen/Fransoo/Norde/Slikker (08), Slikker/Fransoo/Wouters (05)

- **Economic lot-sizing**
  - Chen/Zhang (16), Gopaladesikan/Uhan/Zou (12), Toriello/Uhan (14), van den Heuvel/Borm/Hamers (07)

- **Inventory routing**
  - Özener/Ergun/Savelsbergh (13)

- **Joint replenishment**
  - He/Zhang/Zhang (12), Zhang (09)
Linear Production Games

Owen (75)

\[ f(U) = \min_{x \geq 0} \; cx \]

s.t. \( Ax = \sum_U d_i \)

Theorem

If \( \hat{\lambda} \) is dual optimal for \( f(N) \), then \( \hat{\chi}_i = \hat{\lambda}_d_i \) for all \( i \in N \) is in the core of \( f \).

Proof.

Strong duality \( \Rightarrow \) efficiency

Weak duality \( \Rightarrow \) stability
Linear Production Games

Owen (75)

\[
f(U) = \min_{x \geq 0} cx \\
\text{s.t. } Ax = \sum_U d_i
\]

**Theorem**

If \( \hat{\lambda} \) is dual optimal for \( f(N) \),

\[
\hat{\chi}_i = \hat{\lambda}d_i, \quad i \in N
\]

is in the core of \( f \).
Linear Production Games
Owen (75)

\[
f(U) = \min_{x \geq 0} cx = \max_{\lambda} \lambda \left( \sum_U d_i \right)
\]

s.t. \( Ax = \sum_U d_i \)

s.t. \( \lambda A \leq c \)

**Theorem**

If \( \hat{\lambda} \) is dual optimal for \( f(N) \),

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\]

is in the core of \( f \).

**Proof.**

strong duality \( \Rightarrow \) efficiency \quad weak duality \( \Rightarrow \) stability
Cooperation under Uncertainty
Dynamic LP games

» Players face two-stage decision process defined by

\[
f^1(U) := \min_{x,s \geq 0} \ c^1 x^1 + h^1 s^1 + \mathbb{E}[c^\tau x^\tau + h^\tau s^\tau] \]

s.t. \(A^1 x^1 - C^1 s^1 = \sum_U d^1_i\) \hspace{1cm} (stage 1)

\(A^t x^t + B^t s^1 - C^t s^t = \sum_U d^t_i, \hspace{0.5cm} t \in D, \hspace{1cm} \text{(stage 2)}\)

\(x\)'s are actions (e.g. orders) and \(s\)'s are states (e.g. inventory).
Cooperation under Uncertainty
Dynamic LP games

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s.t.

\[
A^1 x^1 - C^1 s^1 = \sum_U d^1_i \quad \text{(stage 1)}
\]

\[
A^t x^t + B^t s^1 - C^t s^t = \sum_U d^t_i, \quad t \in \mathcal{D}, \quad \text{(stage 2)}
\]

x’s are actions (e.g. orders) and s’s are states (e.g. inventory).

- Static core does not capture timing, need dynamic allocation.
  - \(\chi^t_i\): What player i pays if t realizes.
Suppose players implement optimal solution $\hat{x}, \hat{s}$.

Strong sequential core (Predtetchinski/Herings/Peters 02): Allocations $\chi$ that are
Cooperation under Uncertainty

Dynamic LP games

- Suppose players implement optimal solution $\hat{x}$, $\hat{s}$.

- Strong sequential core (Predtetchinski/Herings/Peters 02):
  Allocations $\chi$ that are
  stage-wise efficient: pay as you go,

$$\sum_{N} \chi^t_i \geq c^t \hat{x}^t + h^t \hat{s}^t, \quad t \in D \cup 1,$$
Cooperation under Uncertainty
Dynamic LP games

- Suppose players implement optimal solution $\hat{x}, \hat{s}$.

- **Strong sequential core** (Predtetchinski/Herings/Peters 02): Allocations $\chi$ that are
  stage-wise efficient: pay as you go,
  \[ \sum_{N} \chi_{i}^{t} \geq c^{t}\hat{x}^{t} + h^{t}\hat{s}^{t}, \quad t \in \mathcal{D} \cup 1, \]
  time-consistently stable: never vulnerable to any $U$ defecting,
  \[ \sum_{U} (\chi_{i}^{1} + \mathbb{E}[\chi_{i}^{T}]) \leq f^{1}(U) \quad \text{(stage 1)} \]
  \[ \sum_{U} \chi_{i}^{t} \leq f^{t}(U, \hat{s}_{U}^{1}), \quad t \in \mathcal{D} \quad \text{(stage 2)} \]

  where $f^{t}$ is static problem faced by $U$ in scenario $t$. 
Cooperation under Uncertainty
Dynamic LP games

Theorem (e.g. Xu/Veinott 13)

A dual-based dynamic allocation similar to Owen’s is in the strong sequential core.

- Similar results extend to multi-stage models.
Example: Newsvendor Games

Inventory pooling

- Each player faces uncertain demand $d_i^t$, can order now, backlog or salvage later:

$$f_{nv}^1(U) := \min_{x,s \geq 0} x^1 + \mathbb{E}[bx^T - vs^T]$$

s.t. $x^1 + x^t - s^t = \sum_U d_i^t, \quad t \in D.$
Example: Newsvendor Games

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$$f_{nv}^1(U) := \min_{x,s \geq 0} x^1 + \mathbb{E}[b x^\tau - v s^\tau]$$

s.t. $x^1 + x^t - s^t = \sum_U d^t_i$, $t \in \mathcal{D}$.

- Suppose scenarios ordered by $\sum N d^t_i$. Then critical index is

$$\hat{t} := \max_{t \in \mathcal{D}} \mathbb{P}(\text{demand} \geq \sum N d^t_i) \geq \frac{1-v}{b-v}.$$ 

Optimal order is $\hat{x}^1 = \sum N d^\hat{t}_i$. 
Example: Newsvendor Games

Inventory pooling

- If $\hat{s}_i^1 = \hat{d}_i^t$, i.e. players split order according to demand in $\hat{t}$,

$$\hat{x}_i^1 = d_i^t, \quad \hat{x}_i^t = \begin{cases} b(d_i^t - \hat{d}_i^t), & t > \hat{t} \\ -v(d_i^t - \hat{d}_i^t), & t < \hat{t} \\ 0, & t = \hat{t} \end{cases}$$

is in the SSC.

- Each player pays to order their demand under scenario $\hat{t}$.
  - Second-stage order/salvage and side payments occur at price $b/v$ depending on realization.
Coherent Risk Measures
Artzner/Delbaen/Eber/Heath (99)

- We now suppose players may be risk-averse.
- Assume risk preferences captured by coherent risk measure, \( \rho : \mathbb{R}^D \to \mathbb{R} \) with
Coherent Risk Measures
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  monotonicity: \( \rho(\chi^T) \leq \rho(\xi^T) \) if \( \chi \leq \xi \).
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Assume risk preferences captured by coherent risk measure, $\rho : \mathbb{R}^D \rightarrow \mathbb{R}$ with

- monotonicity: $\rho(\chi^T) \leq \rho(\xi^T)$ if $\chi \leq \xi$,
- translation invariance: $\rho(\chi^T + \kappa) = \rho(\chi^T) + \kappa$ for constant $\kappa$, (convexity)
We now suppose players may be risk-averse.

Assume risk preferences captured by coherent risk measure, $\rho : \mathbb{R}^D \to \mathbb{R}$ with

- **monotonicity:** $\rho(\chi^\tau) \leq \rho(\xi^\tau)$ if $\chi \leq \xi$,
- **translation invariance:** $\rho(\chi^\tau + \kappa) = \rho(\chi^\tau) + \kappa$ for constant $\kappa$,
- **positive homogeneity:** $\rho(\kappa \chi^\tau) = \kappa \rho(\chi^\tau)$ for constant $\kappa \geq 0$. 

Expectation is risk-neutral, additive instead of subadditive.
We now suppose players may be risk-averse.

Assume risk preferences captured by **coherent risk measure**, \( \rho : \mathbb{R}^D \rightarrow \mathbb{R} \) with

- **monotonicity**: \( \rho(\chi^T) \leq \rho(\xi^T) \) if \( \chi \leq \xi \),
- **translation invariance**: \( \rho(\chi^T + \kappa) = \rho(\chi^T) + \kappa \) for constant \( \kappa \),
- **positive homogeneity**: \( \rho(\kappa \chi^T) = \kappa \rho(\chi^T) \) for constant \( \kappa \geq 0 \),
- **subadditivity**: \( \rho(\chi^T + \xi^T) \leq \rho(\chi^T) + \rho(\xi^T) \) (convexity)

Expectation is risk-neutral, additive instead of subadditive.
Coherent Risk Measures
Artzner/Delbaen/Eber/Heath (99)

Theorem
Any coherent risk measure has a robust representation as a worst-case expectation over a closed, convex set of distributions,

$$\rho(\chi^T) = \max_{q \in Q} \mathbb{E}_q[\chi^T]$$

for $Q \subseteq \Delta^D$.

- Our results extend to multi-stage models using conditional risk mappings (Ruszczyński/Shapiro 06).
Including Risk Aversion in LP Game

- Suppose players face same situation.
  - Linear dynamics, additive demand/requirements, two stages.
- But each player $i$ assesses uncertain costs based on coherent risk measure $\rho_i$.
  - What solution should they implement?
  - Given a solution, how are costs split? What do players think is fair?
The Risk-Averse Strong Sequential Core

If the players implement solution $\hat{x}$, $\hat{s}$, the risk-averse SSC has allocations $\chi$ with
The Risk-Averse Strong Sequential Core

If the players implement solution $\hat{x}$, $\hat{s}$, the risk-averse SSC has allocations $\chi$ with

stage-wise efficiency:

$$\sum_{N} \chi_i^t \geq c^t \hat{x}^t + h^t \hat{s}^t, \quad t \in \mathcal{D} \cup 1,$$
The Risk-Averse Strong Sequential Core

If the players implement solution \( \hat{x}, \hat{s} \), the risk-averse SSC has allocations \( \chi \) with

stage-wise efficiency:

\[
\sum_{N} \chi_{i}^{t} \geq c^{t} \hat{x}^{t} + h^{t} \hat{s}^{t}, \quad t \in D \cup 1,
\]

time-consistent stability: for coalition \( U \subseteq N \),

stage 1 No other solution \( x', s' \) satisfies \( U \)'s demand
and has a stage-wise efficient allocation \( \xi \) with

\[
\xi_{i}^{1} + \rho_{i}(\xi_{i}^{T}) < \chi_{i}^{1} + \rho_{i}(\chi_{i}^{T}), \quad i \in U,
\]

stage 2 \( \sum_{U} \chi_{i}^{t} \leq f^{t}(U, \hat{s}_{U}^{1}), \quad t \in D \).
The Risk-Averse Strong Sequential Core

Theorem

The SSC for solution \( \hat{x}, \hat{s} \) is the set of allocations \( \chi \) with

\[
\sum_{N} \chi_t^i \geq c^t \hat{x}^t + h^t \hat{s}^t, \quad t \in D \cup 1, \quad \text{(efficiency)}
\]

\[
\sum_U \left( \chi^1_i + \rho_i(\chi^T_i) \right) \leq f^1(U), \quad U \subseteq N \quad \text{(stage-1 stable)}
\]

\[
\sum_U \chi_t^i \leq f^t(U, \hat{s}_U^1), \quad U \subseteq N, t \in D, \quad \text{(stage-2 stable)}
\]
The Risk-Averse Strong Sequential Core

**Theorem**

The SSC for solution $\hat{x}$, $\hat{s}$ is the set of allocations $\chi$ with

$$
\sum N \chi_i^t \geq c^t \hat{x}^t + h^t \hat{s}^t, \quad t \in \mathcal{D} \cup 1, \quad \text{(efficiency)}
$$

$$
\sum U (\chi_i^1 + \rho_i(\chi_i^\tau)) \leq f^1(U), \quad U \subseteq N \quad \text{(stage-1 stable)}
$$

$$
\sum U \chi_i^t \leq f^t(U, \hat{s}^1_U), \quad U \subseteq N, t \in \mathcal{D}, \quad \text{(stage-2 stable)}
$$

where

$$
f^1(U) := \min_{x,s \geq 0; \xi} \sum_U (\xi_i^1 + \rho_i(\xi_i^\tau))
$$

s.t. $A^1 x^1 - C^1 s^1 = \sum U d_i^1$

$$
A^t x^t + B^t s^1 - C^t s^t = \sum U d_i^t, \quad t \in \mathcal{D},
$$

$$
\sum U \xi_i^t \geq c^t x^t + h^t s^t, \quad t \in \mathcal{D} \cup 1.
$$
The Risk-Averse Strong Sequential Core

Structural implications

If the players implement $\hat{x}$, $\hat{s}$ and use allocation $\hat{\chi}$ in the SSC,
risk optimality: $\hat{x}$, $\hat{s}$ and $\hat{\chi}$ are optimal for $f^1(N)$,

$$\sum_N (\hat{\chi}_i^1 + \rho_i(\hat{\chi}_i^T)) = f^1(N),$$

convexity: if the risk measures $\rho_i$ are convex, so is the SSC.
Corollary

If $\hat{\chi}$ is in the SSC,

$$\sum_N \rho_i(\hat{\chi}_i^T) = \sum_N \mathbb{E}_{\hat{q}} [\hat{\chi}_i^T],$$

for some $\hat{q} \in \Delta^D$. 

▶ The allocation must be risk-aligned; the same distribution must yield worst-case expected cost for all players. 
▶ E.g. player A cannot hope for low demand if player B hopes for high demand.
Corollary

If \( \hat{\chi} \) is in the SSC,

\[
\sum_{i} N \rho_i(\hat{\chi}_i^T) = \sum_{i} \mathbb{E}_{\hat{q}}[\hat{\chi}_i^T],
\]

for some \( \hat{q} \in \Delta^D \).

- The allocation must be risk-aligned; the same distribution must yield worst-case expected cost for all players.
- E.g. player A cannot hope for low demand if player B hopes for high demand.
Consequences for Cooperation
Pooling versus stability

Corollary

If player $j$ is equally or less risk-averse than all others,

$$Q_j \subseteq Q_i, \quad i \in N \setminus j,$$

then $f^1$ can be optimized by assigning all cost to $j$. 
Consequences for Cooperation
Pooling versus stability

Corollary

If player $j$ is equally or less risk-averse than all others,

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then $f^1$ can be optimized by assigning all cost to $j$.

- Optimization requires the pooling of cost (subadditivity),
  stability requires this cost spread out without increasing risk.
Corollary

If player $j$ is equally or less risk-averse than all others,

$$Q_j \subseteq Q_i, \quad i \in N \setminus j,$$

then $f^1$ can be optimized by assigning all cost to $j$.

- Optimization requires the pooling of cost (subadditivity), stability requires this cost spread out without increasing risk.
- May only be possible by assigning all uncertainty to $j$, while other players get scenario-independent side payments.
Consequences for Cooperation

Compatible beliefs

Corollary

The SSC is empty if

\[ \bigcap_{N} Q_i = \emptyset. \]

- If players cannot agree on a common belief (distribution) about the future, they cannot cooperate.
Consequences for Cooperation
Compatible beliefs

Corollary
*The SSC is empty if*

\[ \bigcap_{N} Q_i = \emptyset. \]

- If players cannot agree on a common belief (distribution) about the future, they cannot cooperate.
- E.g. player A believes tomorrow will be sunny, B believes it will be cloudy. Each can bet arbitrary amount on their belief to create artificial risk arbitrage.
Assume players assess risk with same comonotonic measure, $\rho$.

$$f_{nv}^1(U) := \min_{x,s \geq 0} x^1 + \rho(bx^T - vs^T)$$

s.t. $x^1 + x^t - s^t = \sum_U d^t_i, \quad t \in D.$

Comonotonicity: $\rho(\chi^T + \xi^T) = \rho(\chi^T) + \rho(\xi^T)$ if $\chi, \xi$ comonotonic.
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Critical scenario index $\hat{t}$ defined with worst-case distribution.

Still optimal to order $\hat{x}^1 = \sum_N d_i^\hat{t}$ in first stage.
Recall solution and allocation with $\hat{s}_i^1 = d_i^\hat{t}$ and

\[ \hat{\chi}_i^1 = d_i^\hat{t}, \quad \hat{\chi}_i^t = \begin{cases} 
  b(d_i^t - d_i^\hat{t}), & t > \hat{t} \\
  -v(d_i^\hat{t} - d_i^t), & t < \hat{t} \\
  0, & t = \hat{t}.
\]
Recall solution and allocation with $\hat{s}_i^1 = d_i^t$ and

$$\hat{\chi}_i^1 = d_i^t, \quad \hat{\chi}_i^t = \begin{cases} b(d_i^t - d_i^t), & t > \hat{t} \\ -v(d_i^t - d_i^t), & t < \hat{t} \\ 0, & t = \hat{t}. \end{cases}$$

**Theorem**

Suppose players’ demand is comonotonic, $d_i^1 \leq d_i^2 \leq \ldots$ for $i \in N$. Then $\hat{\chi}$ is in the SSC.

- If demand is comonotonic, players perceive least benefit from cooperation.
  - $\hat{\chi}$ mirrors what each player would incur on their own.
Risk-Averse Newsvendor Games
Bad news: an example with empty SSC

- Two players, exactly complementary demand,

\[ d_{10}^{01} = d_{20}^{10} = 0, \quad d_{10}^{10} = d_{20}^{01} = 1. \]

Ideal situation: Optimal joint order is 1, no uncertainty.

- Risk measure defined by \( Q = \{ (q, 1 - q), (1 - q, q) \} \).
Two players, exactly complementary demand,

\[ d_{10}^{1} = d_{21}^{1} = 0, \quad d_{11}^{1} = d_{20}^{1} = 1. \]

Ideal situation: Optimal joint order is 1, no uncertainty.

Risk measure defined by \( Q = \{(q, 1 - q), (1 - q, q)\}. \)

If \( v > 0, b \) big enough, \( q \neq 1/2 \), the SSC is empty.
Risk-Averse Newsvendor Games
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Ideal situation: Optimal joint order is 1, no uncertainty.

▶ Risk measure defined by \( Q = \{(q, 1 - q), (1 - q, q)\} \).

▶ If \( v > 0, b \) big enough, \( q \neq 1/2 \), the SSC is empty.

▶ No matter how players split the order, the SSC requires the “losing” player to receive \( v \) per unit, but the “winning” player to not pay anything.

▶ Holds for any \( q \neq 1/2 \), any risk measure that isn’t risk-neutral.
Conclusions

- In dynamic LP games, cooperation always possible when players are risk neutral.

- When players are risk-averse, cooperation much more difficult.
  - SSC can easily be empty, implying cooperation unlikely.
  - We can only guarantee SSC is non-empty when cooperation is least beneficial.

- Results suggest risk neutrality is important necessary condition in cooperation.
  - Risk aversion may explain lack of cooperation in situations where risk-neutral models predict it.