Dynamic LP Games with Risk-Averse Players

Alejandro Toriello
Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology

Decision Sciences Seminar
Fuqua School of Business, Duke University
September 21, 2016

joint with Nelson Uhan (USNA)
Sharing and Risk Aversion Don’t Mix: Dynamic LP Games with Risk-Averse Players

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Motivation
California cut flower industry (Nguyen/Toriello/Dessouky/Moore 13)

- Study of potential consolidation center estimated up to 1/3 reduction in transport costs. Issue deemed existential by some in industry.
  - Results presented before U.S. Congress.
  - Application submitted to USDOT for discretionary grant.
Motivation

California cut flower industry (Nguyen/Toriello/Dessouky/Moore 13)

- Study of potential consolidation center estimated up to 1/3 reduction in transport costs. Issue deemed existential by some in industry.
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- However, many farmers skeptical of consolidation and wary of potential risks.

- **Looming question:** Can many disparate agents agree on cooperation and cost sharing? How does aversion to risk affect the ability to cooperate?
Outline

Introduction
- A Crash Course in Cooperative Games
- Primer on Coherent Risk Measures

Model Formulation
- The Risk-Averse Strong Sequential Core
- Implications on Cooperation
- Application: Risk-Averse Newsvendor Games

Parting Thoughts
Cost Allocation in OR Models

(Static) cooperative games and the core

- Players: \( N = \{1, \ldots, n\} \), e.g. retailers or producers.

- Cost function: \( f : 2^N \to \mathbb{R} \), e.g. joint venture cost when some subset of players cooperates.
Cost Allocation in OR Models
(Static) cooperative games and the core

- Players: $N = \{1, \ldots, n\}$, e.g. retailers or producers.

- Cost function: $f : 2^N \rightarrow \mathbb{R}$, e.g. joint venture cost when some subset of players cooperates.

- Goal: Assuming all players in $N$ cooperate, find a cost allocation $\chi \in \mathbb{R}^N$ that splits cost “fairly.”

- Usually models situations in which agents enter into binding agreement.
  - Especially when cooperation occurs over time.
Cost Allocation in OR Models

Cooperative games and the core

- **Core** (Gillies 59): Set of allocations $\chi$ that are
  
  efficient: $\sum_N \chi_i \geq f(N)$,

  stable: for $U \subseteq N$, no allocation $\xi$ has

  $\sum_U \xi_i \geq f(U)$ and $\xi_i < \chi_i$, $i \in U$.

  No player or coalition can do better by *defecting*. 
Cost Allocation in OR Models
Cooperative games and the core

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  efficient: $\sum_N \chi_i \geq f(N)$,

  stable: for $U \subseteq N$, no allocation $\xi$ has

  $$\sum_U \xi_i \geq f(U) \quad \text{and} \quad \xi_i < \chi_i, \; i \in U.$$ 

  No player or coalition can do better by *defecting*.

- Equivalently described by

  $$\{ \chi \in \mathbb{R}^N : \sum_N \chi_i \geq f(N); \; \sum_U \chi_i \leq f(U), U \subseteq N \}. $$

- Non-empty core suggests cooperation is possible.
Some Application Examples
Production, inventory and supply chain management

- Newsvendor and inventory centralization.
  - Chen (09), Chen/Zhang (09), Hartman/Dror (96,05), Hartman/Dror/Shaked (00), Montrucchio/Scarsini (07), Müller/Scarsini/Shaked (02), Özen/Fransoo/Norde/Slikker (08), Slikker/Fransoo/Wouters (05)

- Economic lot-sizing
  - Chen/Zhang (16), Gopaladesikan/Uhan/Zou (12), Toriello/Uhan (14), van den Heuvel/Borm/Hamers (07)

- Inventory routing
  - Özener/Ergun/Savelsbergh (13)

- Joint replenishment
  - He/Zhang/Zhang (12), Zhang (09)
Linear Production Games
Owen (75)

\[
f(U) = \min_{x \geq 0} cx \\
\text{s.t. } Ax = \sum_U d_i
\]
Linear Production Games
Owen (75)

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\[ \text{s.t. } Ax = \sum_U d_i \]

Theorem
If \( \hat{\lambda} \) is dual optimal for \( f(N) \),

\[ \hat{\chi}_i = \hat{\lambda}d_i, \quad i \in N \]

is in the core of \( f \).
Linear Production Games

Owen (75)

\[ f(U) = \min_{x \geq 0} cx = \max_{\lambda} \lambda \left( \sum_U d_i \right) \]

s.t. \( Ax = \sum_U d_i \quad \text{s.t. } \lambda A \leq c \)

**Theorem**

If \( \hat{\lambda} \) is dual optimal for \( f(N) \),

\[ \hat{\chi}_i = \hat{\lambda} d_i, \quad i \in N \]

is in the core of \( f \).

**Proof.**

strong duality \( \Rightarrow \) efficiency \quad weak duality \( \Rightarrow \) stability
Players face two-stage decision process defined by

\[ f^1(U) := \min_{x,s \geq 0} c^1 x^1 + h^1 s^1 + \mathbb{E}[c^\tau x^\tau + h^\tau s^\tau] \]

s.t. \[ A^1 x^1 - C^1 s^1 = \sum_U d^1_i \] (stage 1)
\[ A^t x^t + B^t s^1 - C^t s^t = \sum_U d^t_i, \quad t \in \mathcal{D}, \] (stage 2)

\(x\)'s are actions (e.g. orders) and \(s\)'s are states (e.g. inventory).
Cooperation under Uncertainty
Dynamic LP games

- Players face two-stage decision process defined by

\[
\begin{align*}
f^1(U) := \min_{x,s \geq 0} & \quad c^1 x^1 + h^1 s^1 + \mathbb{E}[c^\tau x^\tau + h^\tau s^\tau] \\
\text{s.t.} & \quad A^1 x^1 - C^1 s^1 = \sum_U d^1_i \\
& \quad A^t x^t + B^t s^1 - C^t s^t = \sum_U d^t_i, \quad t \in \mathcal{D},
\end{align*}
\]

(Stage 1)

(Stage 2)

\(x\)'s are actions (e.g. orders) and \(s\)'s are states (e.g. inventory).

- Static core does not capture timing, need dynamic allocation.
  - \(\chi^t_i\): What player \(i\) pays if \(t\) realizes.
Cooperation under Uncertainty

Dynamic LP games

- Suppose players implement optimal solution $\hat{x}, \hat{s}$.

- **Strong sequential core** (Predtetchinski/Herings/Peters 02): Allocations $\chi$ that are
Cooperation under Uncertainty
Dynamic LP games

- Suppose players implement optimal solution $\hat{x}$, $\hat{s}$.

- **Strong sequential core** (Predtetchinski/Herings/Peters 02):
  Allocations $\chi$ that are
  stage-wise efficient: pay as you go,

  $$\sum_N \chi_i^t \geq c^t \hat{x}^t + h^t \hat{s}^t, \quad t \in \mathcal{D} \cup 1,$$
Cooperation under Uncertainty
Dynamic LP games

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▶ **Strong sequential core** (Predtetchinski/Herings/Peters 02):
Allocations \( \chi \) that are
stage-wise efficient: pay as you go,

\[
\sum_N \chi_i^t \geq c^t \hat{x}^t + h^t \hat{s}^t, \quad t \in D \cup 1,
\]

time-consistently stable: never vulnerable to any \( U \) defecting,

\[
\sum_U (\chi_i^1 + \mathbb{E}[\chi_i^T]) \leq f^1(U) \quad \text{(stage 1)}
\]
\[
\sum_U \chi_i^t \leq f^t(U, \hat{s}_U^1), \quad t \in D \quad \text{(stage 2)}
\]

where \( f^t \) is static problem faced by \( U \) in scenario \( t \).
Cooperation under Uncertainty
Dynamic LP games

Theorem (e.g. Xu/Veinott 13)

A dual-based dynamic allocation similar to Owen’s is in the strong sequential core.

▶ Similar results extend to multi-stage models.
Example: Newsvendor Games
Inventory pooling

- Each player faces uncertain demand $d^t_i$, can order now, backlog or salvage later:

$$f^1_{nv}(U) := \min_{x, s \geq 0} x^1 + \mathbb{E}[bx^T - vs^T]$$

s.t. $x^1 + x^t - s^t = \sum_U d^t_i, \quad t \in \mathcal{D}$. 
Example: Newsvendor Games

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f^{1}_{nv}(U) := \min_{x,s \geq 0} \ x^1 + \mathbb{E}[bx^T - vs^T] \]
\[\text{s.t. } x^1 + x^t - s^t = \sum_U d^t_i, \quad t \in \mathcal{D}.
\]

- Suppose scenarios ordered by $\sum_N d^t_i$. Then critical index is

\[
\hat{t} := \max_{t \in \mathcal{D}} \mathbb{P}(\text{demand} \geq \sum_N d^t_i) \geq \frac{1-v}{b-v}.
\]

Optimal order is $\hat{x}^1 = \sum_N d^\hat{t}_i$. 
Example: Newsvendor Games
Inventory pooling

- If $\hat{s}^1_i = \hat{d}_i^t$, i.e. players split order according to demand in $\hat{t}$,

$$\hat{\chi}_i^1 = d_i^t, \quad \hat{\chi}_i^t = \begin{cases} b(d_i^t - \hat{d}_i), & t > \hat{t} \\ -v(d_i^t - \hat{d}_i), & t < \hat{t} \\ 0, & t = \hat{t} \end{cases}$$

is in the SSC.

- Each player pays to order their demand under scenario $\hat{t}$.
  - Second-stage order/salvage and side payments occur at price $b/v$ depending on realization.
We now suppose players may be risk-averse.

Assume risk preferences captured by coherent risk measure, $ho : \mathbb{R}^D \to \mathbb{R}$ with
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Assume risk preferences captured by coherent risk measure, \( \rho : \mathbb{R}^D \rightarrow \mathbb{R} \) with

- monotonicity: \( \rho(\chi^T) \leq \rho(\xi^T) \) if \( \chi \leq \xi \),
Coherent Risk Measures
Artzner/Delbaen/Eber/Heath (99)

- We now suppose players may be risk-averse.
- Assume risk preferences captured by coherent risk measure, \( \rho : \mathbb{R}^D \to \mathbb{R} \) with
  
  **monotonicity:** \( \rho(\chi^T) \leq \rho(\xi^T) \) if \( \chi \leq \xi \),
  
  **translation invariance:** \( \rho(\chi^T + \kappa) = \rho(\chi^T) + \kappa \) for constant \( \kappa \),
We now suppose players may be risk-averse.

Assume risk preferences captured by coherent risk measure, \( \rho : \mathbb{R}^D \rightarrow \mathbb{R} \) with

- monotonicity: \( \rho(\chi^T) \leq \rho(\xi^T) \) if \( \chi \leq \xi \),
- translation invariance: \( \rho(\chi^T + \kappa) = \rho(\chi^T) + \kappa \) for constant \( \kappa \),
- positive homogeneity: \( \rho(\kappa \chi^T) = \kappa \rho(\chi^T) \) for constant \( \kappa \geq 0 \),
Coherent Risk Measures
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▶ Assume risk preferences captured by coherent risk measure, \( \rho : \mathbb{R}^D \rightarrow \mathbb{R} \) with

  - monotonicity: \( \rho(\chi^T) \leq \rho(\xi^T) \) if \( \chi \leq \xi \),
  - translation invariance: \( \rho(\chi^T + \kappa) = \rho(\chi^T) + \kappa \) for constant \( \kappa \),
  - positive homogeneity: \( \rho(\kappa \chi^T) = \kappa \rho(\chi^T) \) for constant \( \kappa \geq 0 \),
  - subadditivity: \( \rho(\chi^T + \xi^T) \leq \rho(\chi^T) + \rho(\xi^T) \) (convexity)

▶ Expectation is risk-neutral, additive instead of subadditive.
Coherent Risk Measures
Artzner/Delbaen/Eber/Heath (99)

Theorem

Any coherent risk measure has a robust representation as a worst-case expectation over a closed, convex set of distributions,

\[ \rho(\chi^T) = \max_{q \in Q} \mathbb{E}_q[\chi^T] \]

for \(Q \subseteq \Delta^D\).

- Our results extend to multi-stage models using conditional risk mappings (Ruszczyński/Shapiro 06).
Including Risk Aversion in LP Game

- Suppose players face same situation.
  - Linear dynamics, additive demand/requirements, two stages.
- But each player $i$ assesses uncertain costs based on coherent risk measure $\rho_i$.
  - What solution should they implement?
  - Given a solution, how are costs split? What do players think is fair?
The Risk-Averse Strong Sequential Core

If the players implement solution $\hat{x}$, $\hat{s}$, the risk-averse SSC has allocations $\chi$ with
The Risk-Averse Strong Sequential Core

If the players implement solution $\hat{x}, \hat{s}$, the risk-averse SSC has allocations $\chi$ with

stage-wise efficiency:

$$\sum_N \chi_i^t \geq c^t \hat{x}^t + h^t \hat{s}^t, \quad t \in D \cup 1,$$
The Risk-Averse Strong Sequential Core

If the players implement solution \( \hat{x}, \hat{s} \), the risk-averse SSC has allocations \( \chi \) with

stage-wise efficiency:

\[
\sum_{N} \chi_{i}^{t} \geq c^{t} \hat{x}^{t} + h^{t} \hat{s}^{t}, \quad t \in \mathcal{D} \cup 1,
\]

time-consistent stability: for coalition \( U \subseteq N \),

stage 1 No other solution \( x', s' \) satisfies \( U \)'s demand

and has a stage-wise efficient allocation \( \xi \) with

\[
\xi_{i}^{1} + \rho_{i}(\xi_{i}^{T}) < \chi_{i}^{1} + \rho_{i}(\chi_{i}^{T}), \quad i \in U,
\]

stage 2 \( \sum_{U} \chi_{i}^{t} \leq f^{t}(U, \hat{s}_{U}^{1}), \quad t \in \mathcal{D}. \)
The Risk-Averse Strong Sequential Core

Theorem

The SSC for solution $\hat{x}, \hat{s}$ is the set of allocations $\chi$ with

$$ \sum_{N} \chi_{i}^{t} \geq c^{t} \hat{x}^{t} + h^{t} \hat{s}^{t}, \quad t \in D \cup 1, \text{ (efficiency)} $$

$$ \sum_{U} \left( \chi_{i}^{1} + \rho_{i}(\chi_{i}^{T}) \right) \leq f^{1}(U), \quad U \subseteq N \text{ (stage-1 stable)} $$

$$ \sum_{U} \chi_{i}^{t} \leq f^{t}(U, \hat{s}_{U}^{1}), \quad U \subseteq N, t \in D, \text{ (stage-2 stable)} $$
The Risk-Averse Strong Sequential Core

Theorem

The SSC for solution $\hat{x}, \hat{s}$ is the set of allocations $\chi$ with

$$
\sum_N \chi_i^t \geq c^t \hat{x}^t + h^t \hat{s}^t, \quad t \in D \cup 1, \quad \text{(efficiency)}
$$

$$
\sum_U (\chi_i^1 + \rho_i(\chi_i^T)) \leq f^1(U), \quad U \subseteq N \quad \text{(stage-1 stable)}
$$

$$
\sum_U \chi_i^t \leq f^t(U, \hat{s}_U^1), \quad U \subseteq N, t \in D, \quad \text{(stage-2 stable)}
$$

where

$$
f^1(U) := \min_{x, s \geq 0; \xi} \sum_U (\xi^1_i + \rho_i(\xi_i^T))
$$

s.t. $A^1 x^1 - C^1 s^1 = \sum_U d_i^1$

$$
A^t x^t + B^t s^1 - C^t s^t = \sum_U d_i^t, \quad t \in D,
$$

$$
\sum_U \xi_i^t \geq c^t x^t + h^t s^t, \quad t \in D \cup 1.
$$
If the players implement $\hat{x}$, $\hat{s}$ and use allocation $\hat{\chi}$ in the SSC, risk optimality: $\hat{x}$, $\hat{s}$ and $\hat{\chi}$ are optimal for $f^1(N)$,

$$\sum_N (\hat{\chi}_i^1 + \rho_i(\hat{\chi}_i^T)) = f^1(N),$$

convexity: if the risk measures $\rho_i$ are convex, so is the SSC.
Consequences for Cooperation

Risk alignment

Corollary

If \( \hat{\chi} \) is in the SSC,

\[
\sum_{N} \rho_i(\hat{\chi}_i^T) = \sum_{N} E_{\hat{q}}[\hat{\chi}_i^T],
\]

for some \( \hat{q} \in \Delta^D \).
Consequences for Cooperation
Risk alignment

Corollary
If \( \hat{\chi} \) is in the SSC,

\[
\sum_N \rho_i(\hat{\chi}_i^T) = \sum_N \mathbb{E}_{\hat{q}}[\hat{\chi}_i^T],
\]

for some \( \hat{q} \in \Delta^D \).

- The allocation must be risk-aligned; the same distribution must yield worst-case expected cost for all players.
- E.g. player A cannot hope for low demand if player B hopes for high demand.
Consequences for Cooperation
Pooling versus stability

Corollary

If player $j$ is less or equally risk-averse than all others,

$$Q_j \subseteq Q_i, \quad i \in N \setminus j,$$

then $f^1$ can be optimized by assigning all cost to $j$. 
Consequences for Cooperation
Poolig versus stability

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If player $j$ is less or equally risk-averse than all others,

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- Optimization requires the pooling of cost (subadditivity),
  stability requires this cost spread out without increasing risk.
Consequences for Cooperation
Pooling versus stability

Corollary

If player $j$ is less or equally risk-averse than all others,

$$Q_j \subseteq Q_i, \quad i \in N \setminus j,$$

then $f^1$ can be optimized by assigning all cost to $j$.

- Optimization requires the pooling of cost (subadditivity), stability requires this cost spread out without increasing risk.
- May only be possible by assigning all uncertainty to $j$, while other players get scenario-independent side payments.
Consequences for Cooperation

Compatible beliefs

Corollary

The SSC is empty if

$$\bigcap_{N} Q_i = \emptyset.$$ 

deep down right

- If players cannot agree on a common belief (distribution) about the future, they cannot cooperate.
Consequences for Cooperation
Compatible beliefs

Corollary

*The SSC is empty if*
\[ \bigcap_{i} Q_i = \emptyset. \]

- If players cannot agree on a common belief (distribution) about the future, they cannot cooperate.
- E.g. player A believes tomorrow will be sunny, B believes it will be cloudy. Each can bet arbitrary amount on their belief to create artificial risk arbitrage.
Assume players assess risk with same comonotonic measure, $\rho$.

$$f_{nv}^1(U) := \min_{x,s \geq 0} x^1 + \rho(bx^T - vs^T)$$

s.t. $x^1 + x^t - s^t = \sum_U d^t_i$, $t \in D$.

comonotonicity: $\rho(\chi^T + \xi^T) = \rho(\chi^T) + \rho(\xi^T)$ if $\chi, \xi$ comonotonic
Assume players assess risk with same comonotonic measure, \( \rho \).

\[
 f_{nv}^1(U) := \min_{x,s \geq 0} x^1 + \rho(bx^\tau - vs^\tau) \\
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\]

Comonotonicity: \( \rho(\chi^\tau + \xi^\tau) = \rho(\chi^\tau) + \rho(\xi^\tau) \) if \( \chi, \xi \) comonotonic

- Critical scenario index \( \hat{t} \) defined with worst-case distribution.
- Still optimal to order \( \hat{x}^1 = \sum_N d_i^{\hat{t}} \) in first stage.
Recall solution and allocation with $\hat{s}_i^1 = d_i^t$ and

$$\hat{\chi}_i^1 = d_i^t, \quad \hat{\chi}_i^t = \begin{cases} 
 b(d_i^t - d_i^t), & t > \hat{t} \\
 -v(d_i^t - d_i^t), & t < \hat{t} \\
 0, & t = \hat{t}.
\end{cases}$$
Risk-Averse Newsvendor Games

(Not so) good news

▶ Recall solution and allocation with $\hat{s}^1_i = d^t_i$ and

$$\hat{\chi}^1_i = d^t_i, \quad \hat{\chi}^t_i = \begin{cases} b(d^t_i - d^{\hat{t}}_i), & t > \hat{t} \\ -v(d^{\hat{t}}_i - d^t_i), & t < \hat{t} \\ 0, & t = \hat{t}. \end{cases}$$

Theorem

Suppose players’ demand is comonotonic, $d^1_i \leq d^2_i \leq \ldots$ for $i \in N$. Then $\hat{\chi}$ is in the SSC.

▶ If demand is comonotomic, players perceive least benefit from cooperation.

▶ $\hat{\chi}$ mirrors what each player would incur on their own.
Two players, exactly complementary demand,

\[ d_{1}^{01} = d_{2}^{10} = 0, \quad d_{1}^{10} = d_{2}^{01} = 1. \]

Ideal situation: Optimal joint order is 1, no uncertainty.

Risk measure defined by \( Q = \{(q, 1-q), (1-q, q)\} \).
Risk-Averse Newsvendor Games
Bad news: an example with empty SSC

▶ Two players, exactly complementary demand,

\[ d_{10} = d_{01} = 0, \quad d_{11} = d_{02} = 1. \]

Ideal situation: Optimal joint order is 1, no uncertainty.

▶ Risk measure defined by \( Q = \{(q, 1-q), (1-q, q)\} \).

▶ If \( v > 0, b \) big enough, \( q \neq 1/2 \), the SSC is empty.
Risk-Averse Newsvendor Games
Bad news: an example with empty SSC

- Two players, exactly complementary demand,
  \[ d_{10}^{01} = d_{20}^{10} = 0, \quad d_{10}^{10} = d_{20}^{01} = 1. \]

  Ideal situation: Optimal joint order is 1, no uncertainty.

- Risk measure defined by \( Q = \{(q, 1-q), (1-q, q)\} \).

- If \( v > 0 \), \( b \) big enough, \( q \neq 1/2 \), the SSC is empty.
  
  - No matter how players split the order, the SSC requires the "losing" player to receive \( v \) per unit, but the "winning" player to not pay anything.
  
  - Holds for any \( q \neq 1/2 \), any risk measure that isn’t risk-neutral.
Conclusions

- In dynamic LP games, cooperation always possible when players are risk neutral.
- When players are risk-averse, cooperation much more difficult.
  - SSC can easily be empty, implying cooperation unlikely.
  - We can only guarantee SSC is non-empty when cooperation is least beneficial.
- Results suggest risk neutrality is important necessary condition in cooperation.
  - Risk aversion may explain lack of cooperation in situations where risk-neutral models predict it.