Dynamic Linear Production Games under Uncertainty

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Motivation

California cut flower industry (Nguyen/Toriello/Dessouky/Moore 13)

• In 1960’s, California controlled 2/3 of U.S. cut flower market.
  • Still accounts for most (4/5) of U.S. production.

• Today, South America (mostly Colombia) has 3/4 of market.
  • California down to 1/5.

• Many causes: Cheap labor, trade agreements, ...
  • **Major factor:** Consolidation facility for South American blooms in Miami allows for economies of scale in transportation.
  • Often cheaper per unit to ship flowers from Bogotá than from Santa Barbara!
Motivation
California cut flower industry (Nguyen/Toriello/Dessouky/Moore 13)

• A similar consolidation center in California could significantly reduce transportation costs.
  • Could cut costs by 1/3 if all California growers participate.

• Political/industry momentum building for consolidation; issue deemed existential by some in industry.
  • Results presented before U.S. Congress.
  • Application submitted to USDOT for discretionary grant.

• Looming question: How would such a center be administered? How would costs be fairly split in an ongoing basis?
Outline

Introduction

Static Allocations

Game Definition and Dynamic Allocation

Ongoing & Future Work
Cost Allocation in OR/MS Models
Cooperative games and the core

- **Players**: $N = \{1, \ldots, n\}$, e.g. retailers or producers.
- **Cost function**: $w : 2^N \to \mathbb{R}$, e.g. joint venture cost when some subset of players cooperates.
- **Goal**: Assuming all players in $N$ cooperate, find a cost allocation $\chi \in \mathbb{R}^N$ that splits cost “fairly.”
- **Some attractive properties**:
  - Efficiency: $\sum_N \chi_i = w(N)$ ("balanced budget").
  - Stability: $\sum_U \chi_i \leq w(U)$ for all $U \subseteq N$.
  - Computational efficiency: E.g. computed in poly-time w.r.t. $n$. 
Cost Allocation in OR/MS Models
Cooperative games and the core

- Core (Gillies 59): Set of efficient, stable allocations,

\[ \{ \chi \in \mathbb{R}^N : \sum_N \chi_i = w(N); \sum_U \chi_i \leq w(U), U \subset N \}. \]

No player or coalition can do better by \textit{defecting}.

- Many other concepts available if core is empty, e.g. \( \alpha \)-core, \( \varepsilon \)-core, nucleolus, Shapley value, ...

- The core is a widely applied cost allocation concept for OR/MS models.
  - Game with non-empty core may indicate cooperation in this setting is possible or likely.
Some Application Examples
Production, inventory and supply chain management

• Inventory centralization.
  • Chen (09), Chen/Zhang (09), Hartman/Dror (96,05), Hartman/Dror/Shaked (00), Montrucchio/Scarsini (07), Müllер/Scarsini/Shaked (02), Özen/Fransoo/Norde/Slikker (08), Slikker/Fransoo/Wouters (05)

• Economic lot-sizing
  • Chen/Zhang (06), Gopaladesikan/Uhan/Zou (12), van den Heuvel/Borm/Hamers (07)

• Inventory routing
  • Özener/Ergun/Savelsbergh (13)

• Joint replenishment
  • He/Zhang/Zhang (12), Zhang (09)
Linear Production Games

Owen (75)

\[ w(U) = \min cx \]
\[ \text{s.t. } Ax = \sum_{i \in U} d^i \]
\[ x \geq 0 \]

**Theorem**

*If \( \lambda \) is dual optimal for \( w(N) \),

\[ \chi_i = \lambda d^i, \quad i \in N \]

*is in the core of \( w \).*

**Proof.**

strong duality \( \Rightarrow \) efficiency \quad weak duality \( \Rightarrow \) stability
Linear Production Games
Owen (75)

- Many applications, particularly in network optimization.
  - Assignment (Shapley/Shubik 71), max flow (Kalai/Zemel 82), min cost spanning tree (Granot/Huberman 81), network synthesis (Tamir 91).

- Connections to other games.
  - Facility location (Goemans/Skutella 04), TSP (Toriello/Uhan 13), general combinatorial optimization (Deng/Ibaraki/Nagamochi 99).

- Much subsequent study.
  - Flåm (02), Granot (86), Samet/Zemel (84), van Gellekom/Potters/Reijnierse/Engel/Tijs (00).
Static Allocations: What can go wrong

A lot-sizing example

- Players are retailers or producers facing uncertain demand.
- Scenario tree $T$ dictates, for scenarios $t \in T$, probabilities and corresponding realizations of
  1. linear ordering cost $c_t$,
  2. holding cost $h_t$, and
  3. each player $i$’s demand $d^t_i$.
- $w(U)$ is optimal expected cost of meeting demand for coalition $U$ over horizon given by $T$.
- Static approach: Allocate optimal expected cost up front.
\( \chi = \{30, 31, 22\} \)

1: \( c_1 = 2, h_1 = 1 \)
\( d^1 = (4, 4, 4) \)

2: \( c_2 = 1, h_2 = 1 \)
\( d^2 = (4, 3, 7) \)

3: \( c_3 = 6, h_3 = 1 \)
\( d^3 = (4, 5, 1) \)

4: \( c_4 = 6, h_4 = 1 \)
\( d^4 = (4, 5, 7) \)

5: \( c_5 = 2, h_5 = 1 \)
\( d^5 = (4, 3, 1) \)

6: \( c_6 = 4, h_6 = 1 \)
\( d^6 = (4, 3, 1) \)
\( \chi = (30, 31.5, 22.5) \)
\( \chi^1 = (19.3, 20.3, 14.4) \)

1: \( c_1 = 2, h_1 = 1 \)
\( d^1 = (4, 4, 4) \)

2: \( c_2 = 1, h_2 = 1 \)
\( d^2 = (4, 3, 7) \)

3: \( c_3 = 6, h_3 = 1 \)
\( d^3 = (4, 5, 1) \)

4: \( c_4 = 6, h_4 = 1 \)
\( d^4 = (4, 5, 7) \)

5: \( c_5 = 2, h_5 = 1 \)
\( d^5 = (4, 3, 1) \)

6: \( c_6 = 4, h_6 = 1 \)
\( d^6 = (4, 3, 1) \)
\[ \chi^1 = (19.3, 20.3, 14.4) \]

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2: \( c_2 = 1, h_2 = 1 \)
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3: \( c_3 = 6, h_3 = 1 \)
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5: \( c_5 = 2, h_5 = 1 \)
\[ d^5 = (4, 3, 1) \]

6: \( c_6 = 4, h_6 = 1 \)
\[ d^6 = (4, 3, 1) \]

\[ \text{inv.} = (4, 3, 3) \]
Static Allocations: What can go wrong

A lot-sizing example

- Static allocation covers expected cost, not guaranteed to match realized cost.
- Unrealistic to expect full payment of multi-period, ongoing endeavor up front.
  - More problematic in infinite horizons and/or when discounting.
- Even core allocations vulnerable to defections mid-way through horizon.
  - Cannot fix with naive, proportional “on-the-fly” allocations.
  - Must account for players’ division of jointly produced resources.
Dynamic Cooperative Games

• Active research area in last decade.
  • Avrachenkov/Cottatellucci/Maggi (13), Elkind/Pasechnik/Zick (13), Gale (78), Habis/Herings (10), Kranich/Perea/Peters (05), Lehrer/Scarsini (13), Petrosjan (02), Predtetchinski (07), Predtetchinski/Herings/Peters (02,04)

• Limited related discussions in OR/MS literature.
  • Chen/Zhang (09), Flåm (02), Nagarajan/Sošić (07), Özener/Ergun/Savelsbergh (13)
Dynamic Cooperative Games

Definition (Strong Sequential Core)

Set of dynamic allocations satisfying:

1. **Stage-wise efficiency**: Costs are covered exactly, as they are incurred.

2. **Time-consistent stability**: At any point in time, any coalition’s expected allocation from that point forward does not exceed its expected cost if defecting.

- Definition taken from Kranich/Perea/Peters (05).
- Conditions imply their static counterparts.
- Other related concepts studied, e.g. *weak* sequential core.
Game Setup and Parameters

- Scenario tree $T$; each scenario $t$ has
  - a parent $a(t)$, representing the immediate past,
  - children $D_t$, the possible immediate future, and
  - probability $p_t$ of occurring.

- At $t$, each player $i$ has
  - an initial state vector $s_{i}^{a(t)}$,
  - actions $x_{i}^t$ with costs $c_{i}^t$,
  - a fixed resource or demand vector $d_{i}^t$,
  - an ending state vector $s_{i}^t$ with costs $h_{i}^t$, determined by $s_{i}^{a(t)}$, $x_{i}^t$, $d_{i}^t$. 
Game Setup and Parameters

• If a coalition $U \subseteq N$ decides to cooperate, each scenario’s ending state is determined by the linear dynamics

$$\sum_{i \in U} A^t_i x^t_i + \sum_{i \in U} B^t_i s^t_{i(a(t) - \sum_{i \in U} C^t_i s^t_i} = \sum_{i \in U} d^t_i,$$

where matrices $A^t_i$, $B^t_i$, $C^t_i$ have appropriate dimension.

• We do not assume the $s^t_{i}$’s are uniquely determined; players may have to choose a set of feasible ending states.
  • E.g. in lot-sizing, players can assign ownership of inventory as they wish.
Game Formulation

Dynamic linear production game under uncertainty

- Assuming scenario \( r \in \mathcal{T} \) realizes and incoming states are \( \hat{s}^a(r) \), if coalition \( U \) forms they solve

\[
\begin{align*}
\omega_r(U, \hat{s}^a(r)) := \\
\min_{x,s} \sum_{t \in \mathcal{T}_r} p_{t|r} \sum_{i \in U} (c^t_i x^t_i + h^t_i s^t_i) \\
\text{s.t.} \quad \sum_{i \in U} A^r_i x^r_i - \sum_{i \in U} C^r_i s^r_i = \sum_{i \in U} d^r_i - \sum_{i \in U} B^r_i \hat{s}^a_i(r), \\
\sum_{i \in U} A^t_i x^t_i + \sum_{i \in U} B^t_i s^a_i(t) - \sum_{i \in U} C^t_i s^t_i = \sum_{i \in U} d^t_i, \quad t \in \mathcal{T}_r \setminus r, \\
x^t_i \geq 0, s^t_i \geq 0, \quad i \in U, t \in \mathcal{T}_r.
\end{align*}
\]

- Note dependence on current scenario and incoming state.
Game Formulation

Dynamic linear production game under uncertainty

- Dual formulation:

\[
\begin{align*}
\omega_r(U, \hat{s}^a(r)) &= \\
\max_{\lambda} \quad & \sum_{t \in T_r} \lambda^t \sum_{i \in U} d_i^t - \lambda^r \sum_{i \in U} B_i^r \hat{s}_i^a(r) \\
\text{s.t.} \quad & \lambda^t A_i^t \leq p_{t|r} c_i^t, \quad i \in U, t \in T_r, \\
& \sum_{\tau \in D_t} \lambda^\tau B_i^\tau - \lambda^t C_i^t \leq p_{t|r} h_i^t, \quad i \in U, t \in T_r.
\end{align*}
\]
Dynamic Allocation

- Allocation $\chi^t_i$: Player $i$ pays this if scenario $t$ is realized.

Definition (Strong Sequential Core)

For an optimal solution $(\hat{x}, \hat{s})$ of $w_1(N, \hat{s}^0)$, an allocation $\chi$ satisfying:

1. Stage-wise efficiency: For any scenario $t$,
   \[
   \sum_{i \in N} \chi^t_i = \sum_{i \in N} (c^t_i \hat{x}^t_i + h^t_i \hat{s}^t_i).
   \]

2. Time-consistent stability: For any scenario $r$ and coalition $U$,
   \[
   \sum_{t \in T_r} \sum_{i \in U} \ p^t_{|r} \chi^t_i \leq w_r(U, \hat{s}^a(r)).
   \]
Dynamic Allocation

Theorem
If $(\hat{x}, \hat{s})$ and $\hat{\lambda}$ are respectively primal and dual optimal for $w_1(N, \hat{s}^0)$, the allocation

$$\hat{\chi}_i^t := \frac{1}{p_t} \left( \hat{\lambda}^t (d_i^t - B_i^t s_i^{a(t)}) + \sum_{\tau \in D_t} \hat{\lambda}^\tau B_i^\tau s_i^\tau \right), \quad i \in N, t \in T$$

is in the strong sequential core.

- Based on optimal dual prices, this allocation
  1. pays for current demand/resources
  2. minus a credit based on previous activity,
  3. and pays for the ending state.
Dynamic Allocation

Proof sketch.

1. Stage-wise efficiency: Follows from complementary slackness and primal feasibility of \((\hat{x}, \hat{s})\).

2. Time-consistent stability: Follows from dual feasibility of \(\hat{\lambda}\).

- Proof similar in spirit to Owen (75), though slightly more complicated.
Dynamic Allocation

Additional properties

• The allocation $\hat{\chi}$ requires access only to primal and dual optimal solutions.
  • Requires no additional effort once optimization is carried out.

• At an encountered scenario $t$, need only information related to $t$ and its children.
  • Localized information need amenable to special-purpose methods, when available.
Infinite-Horizon Game

- Same construction can define an infinite-horizon game.
  - \( \mathcal{T} \) has infinite height, but need every scenario to have finitely many children.
  - Assume optimal solutions have finite cost, e.g. via discounting.
  - Minor additional technical conditions.

**Theorem**

*Under these conditions, the dynamic allocation \( \hat{\chi} \) is in the strong sequential core of the infinite-horizon dynamic linear production game under uncertainty.*

**Proof sketch.**

Follows from finite case and strong duality, complementarity results of Romeijn/Smith (98) for countably infinite LP’s.
Infinite-Horizon Game

Another lot-sizing example (Romeijn/Smith 98)

- Deterministic, infinite-horizon version of linear lot-sizing model.

\[
\begin{align*}
\min_{x,s} & \quad \sum_{t=1}^{\infty} \sum_{i \in N} (c^t x^t_i + h^t s^t_i) \\
\text{s.t.} & \quad \sum_{i \in N} x^1_i - \sum_{i \in N} s^1_i = \sum_{i \in N} d^1_i, \\
& \quad \sum_{i \in N} x^t_i + \sum_{i \in N} s^{t-1}_i - \sum_{i \in N} s^t_i = \sum_{i \in N} d^t_i, \quad t = 2, \ldots \\
& \quad x^t_i \geq 0, s^t_i \geq 0, \quad i \in N, \quad t = 1, \ldots .
\end{align*}
\]
Infinite-Horizon Game
Another lot-sizing example (Romeijn/Smith 98)

- Dual of lot-sizing problem is

\[
\max_{\lambda} \sum_{t=1}^{\infty} \lambda^t \sum_{i \in N} d_i^t \\
\text{s.t. } \lambda^t \leq c^t \quad \text{for } t = 1, \ldots \\
\phantom{\text{s.t. } } \lambda^{t+1} - \lambda^t \leq h^t \quad \text{for } t = 1, \ldots
\]

- Dual optimal solution is

\[
\hat{\lambda}^1 = c^1 \\
\hat{\lambda}^{t+1} = \min\{c^{t+1}, h^t + \hat{\lambda}^t\}, \quad t = 1, \ldots
\]

i.e. pick cheaper of two options, produce now or hold over from last period.
Infinite-Horizon Game

Another lot-sizing example (Romeijn/Smith 98)

- Extreme primal optimal solutions exhibit replenishment interval structure.
- Dynamic allocation is

\[ \hat{\chi}_i^t = \hat{\lambda}^t (d_i^t - \hat{s}_i^{t-1}) + \hat{\lambda}^{t+1} \hat{s}_i^t. \]

In period \( t \), player \( i \) pays \( \hat{\lambda}^t \) for each unit of his demand, minus incoming inventory (paid previously), and pays \( \hat{\lambda}^{t+1} \) per unit of ending inventory.
Conclusions

- Many cooperative endeavors in OR/MS applications must be carried out over a horizon of multiple periods, perhaps indefinitely.
  - Static cost allocation concepts not adequate.

- Introduced dynamic linear production games under uncertainty, which can model many such applications.
  - Constructed allocation in *strong sequential core*, the natural generalization of the core.

- More generally, hope to introduce notions of stage-wise efficiency and time-consistent stability to dynamic, cooperative OR/MS models.
Ongoing Work

Non-linear models

• Of course, many models require non-linearity in cost or structure.
  • But general concept still applies.

• Toriello/Uhan (13): Constructed dynamic allocation in strong sequential core for deterministic lot-sizing model with concave order costs.
  • Construction leverages previous work on static allocations (Chen/Zhang 06, Gopaladesikan/Uhan/Zou 12).
  • Similar approach may work in other applications.
Ongoing Work

Risk aversion

- Models allocating uncertain costs usually assume objective of minimizing expected costs, i.e. assume players and coalitions are risk-neutral.
  - Can we incorporate different attitudes towards risk?
  - Require employing multi-period risk measures.

- Our construction and proof do not carry over even to min-max (most risk-averse) case.
  - Different coalitions may see different scenarios as worst cases.
Ongoing Work

Risk aversion

- Uhan (13): Core allocations for two-stage linear programming game, where each player’s objective is a coherent risk measure.
  - Requires more general, Pareto definition of stability.
  - “Core equivalence” between original game and static, deterministic game defined by risk measures.

- Unclear which risk measures can be employed in multi-period settings.
  - Even coherent risk measures problematic.