Cooperative Traveling Salesman Games with Asymmetric Arc Costs

Alejandro Toriello

Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology

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Motivation

How much does each retailer pay for the truck?

- Important question when retailers/suppliers are small and independent.
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Cooperative Cost Game

- Players: $N = \{1, \ldots , n\}$; e.g. retail locations.

- Cost function: $C : 2^N \rightarrow \mathbb{R}$; e.g. cost of visiting some subset from depot.

- Goal: Cost allocation $\chi \in \mathbb{R}^N$ that splits cost “fairly.”

Some attractive properties:

- Efficiency: $\chi(N) := \sum_N \chi_i = C(N)$ (“balanced budget”).
- Stability: $\chi(S) \leq C(S')$ for all $S \subseteq N$.
- Computational efficiency: Computed in poly-time w.r.t. $n$. 
The Core

- **Core (Gillies 59):** Set of efficient, stable allocations,

\[
\{\chi \in \mathbb{R}^N : \chi(N) = C(N); \chi(S) \leq C(S), S \subseteq N\}.
\]

No player or coalition can do better by leaving.

- **\(\alpha\)-Core (Faigle/Kern 93):** Allocations are stable but \(\alpha\)-budget-balanced, i.e.

\[
\chi(N) \geq \alpha C(N), \quad \alpha \in (0, 1].
\]

If stability implies budget deficit, external party (e.g. government) may have to subsidize cooperation.

- Many other concepts available if core is empty, e.g. \(\varepsilon\)-core, nucleolus, Shapley value, ...
TSP Games

- TSP Game: \( C(S) \) is cost of optimal TSP tour through \( S \cup \emptyset \).
  - Assume arc costs are non-negative, satisfy triangle inequality: \( c_{ij} + c_{jk} \geq c_{ik} \).

- Faigle/Fekete/Hochstättler/Kern (98):
  - TSP games can have empty core for \( n \geq 6 \).
  - Allocation in \( \frac{2}{3} \)-core for symmetric costs, using LP relaxation.

- Bläser/Shankar Ram (08): Allocation in \( \Omega\left(\frac{1}{\log n}\right) \)-core for asymmetric costs.

- Our contribution:
  - Allocation in \( \Omega\left(\frac{\log \log n}{\log n}\right) \)-core for asymmetric costs, generalizing Faigle et al (98).
  - Simplified proof using linear production games (Owen 75).
  - Example that shows more is possible.
Asymmetric TSP Formulations

\[ C(S) = \min \ cx \]

s.t. \[ x(\delta^+_S(i)) = 1, \quad i \in S \cup 0 \] (enter once)

\[ x(\delta^-_S(i)) = 1, \quad i \in S \cup 0 \] (exit once)

\[ x(\delta^+_S(U)) \geq 1, \quad \emptyset \neq U \subsetneq S \cup 0 \] (subtour elimination)

\[ x \text{ binary,} \]

where \[ \delta^+_S(U) = \{(i, j) : i \in U, j \in S \cup 0 \setminus U\}. \]
Asymmetric TSP Formulations

\[ C_{LP}(S) = \min cx \]

s.t. \[ x(\delta^+(i)) = 1, \quad i \in S \cup 0 \quad \text{(enter once)} \]
\[ x(\delta^-(i)) = 1, \quad i \in S \cup 0 \quad \text{(exit once)} \]
\[ x(\delta^+_S(U)) \geq 1, \quad \emptyset \neq U \subsetneq S \cup 0 \quad \text{(subtour elimination)} \]
\[ x \geq 0 \]

Theorem (A/G/M/OG/S 10)

\[ C_{LP}(S) = \Omega \left( \frac{\log \log n}{\log n} \right) C(S) \]

- Approach: Construct allocation from dual optimal solution of \( C_{LP}(N) \), which is poly-time solvable.
Asymmetric TSP Formulations

The Parsimonious Property

Lemma (Goemans/Bertsimas 93; Nguyen 11)

\[ C_{LP}(S) = \min \ cx \]
\[ s.t. \ x(\delta_S^+(i)) = x(\delta_S^-(i)), \quad i \in S \cup 0 \quad \text{(flow balance)} \]
\[ x(\delta_S^+(U)) \geq 1, \quad \emptyset \neq U \subsetneq S \cup 0 \quad \text{(subtour elimination)} \]
\[ x \geq 0. \]

- Intuition: Min-cost fractional Eulerian graphs are automatically \textit{parsimonious}; i.e. degrees as small as possible.
Asymmetric TSP Formulations

A new relaxation

Lemma

\[ C_{LP}(S) = \min cx \]

\[ s.t. \quad x(\delta_S^+(i)) = 1, \quad i \in S \]

\[ x(\delta_S^-(i)) = 1, \quad i \in S \]

\[ x(\delta_S^+(U)) \geq 1, \quad \emptyset \neq U \subseteq S \]

\[ x \geq 0. \]

- Restrict constraints to players (ignore depot).

Proof sketch.

\[ C_{LP} = \text{parsimonious LP} \leq \text{new LP} \leq C_{LP}. \]
Detour to Linear Production Games

Owen (75)

\[ \tilde{C}(S) = \min cx \]

s.t. \( Bx = \sum_{j \in S} b^j \)

\[ Dx \geq \sum_{j \in S} d^j \]

\[ x \geq 0 \]

**Theorem**

If \( \lambda, \mu \) are dual optimal for \( =, \geq \) constraints in \( \tilde{C}(N) \),

\[ \chi_i = b^i \lambda + d^i \mu, \quad i \in N \]

is in the core of \( \tilde{C} \).

**Proof.**

strong duality \( \Rightarrow \) efficiency, \quad weak duality \( \Rightarrow \) stability.
$C_{LP}$ is a Linear Production Game

Lemma

\[ C_{LP}(S) = \min cx \]
\[ s.t. \ x(\delta^+_N(i)) = \sum_{j \in S} b^j_i, \quad i \in N \]
\[ x(\delta^-_N(i)) = \sum_{j \in S} b^j_i, \quad i \in N \]
\[ x(\delta^+_N(U)) \geq \sum_{j \in S} d^j_{i,U}, \quad \emptyset \neq U \subseteq N, \quad i \in U \]
\[ x \geq 0, \]

where \[ b^j_i = \begin{cases} 1, & i = j \\ 0, & \text{o.w.} \end{cases}, \quad d^j_{i,U} = \begin{cases} 1, & i = j \\ 0, & \text{o.w.} \end{cases}. \]

- Idea: Replicate subtour constraints but only “activate” those for players in coalition \( S \).
- Reminiscent of prize-collecting TSP (Balas 89).
Cost Allocation

• Recall “unreplicated”, player-only formulation:

\[ C_{LP}(N) = \min cx \]
\[ \text{s.t. } x(\delta^+_N(i)) = 1, \ i \in N \]
\[ x(\delta^-_N(i)) = 1, \ i \in N \]
\[ x(\delta^+_N(U)) \geq 1, \ \emptyset \neq U \subseteq N \]
\[ x \geq 0. \]

**Theorem**

If (i) \((\lambda^+, \lambda^-)\), \(\mu\) are dual optimal for \(=, \geq\) constraints and (ii) for each \(i \in U\) we give player “weights” \(\gamma_{i,U} \geq 0\) with \(\sum_{i \in U} \gamma_{i,U} = 1\),

\[ \chi_i = \lambda^+_i + \lambda^-_i + \sum_{U \ni i} \gamma_{i,U} \mu_U \]

is in the core of \(C_{LP}\), and thus in the \(\Omega\left(\frac{\log \log n}{\log n}\right)\)-core of \(C\).
Cost Allocation

Proof.
The “replicated” LP has $|U|$ copies of each subtour constraint, so we can divide the unreplicated constraint’s multiplier $\mu_U$ among $U$’s players any way we wish. The proof then follows from Owen’s argument.

- Significant freedom to design the allocation. E.g.,
  1. Uniform to all players: $\gamma_{i,U} = \frac{1}{|U|}$.
  2. Preferential order: Define ordering of $N$, set $\gamma_{i,U} = 1$ if $i$ is minimal in $U$, 0 otherwise.
A Geometric Example

Jünger & Pulleyblank (93)
A Geometric Example

Cost Allocation

\[ \chi_1 = \lambda_1^+ + \lambda_1^- + \frac{\mu_{12}}{2} \]
\[ \chi_2 = \lambda_2^+ + \lambda_2^- + \frac{\mu_{12}}{2} \]
Extension to Network Design Games

- Strongly $k$-connected Eulerian graph game:

$$C_k(S) = \min cx$$

s.t. $x(\delta^+_S(i)) = k, \quad i \in S \cup 0$

$$x(\delta^-_S(i)) = k, \quad i \in S \cup 0$$

$$x(\delta^+_S(U)) \geq k, \quad \emptyset \neq U \subset S \cup 0$$

$$x \geq 0 \text{ integer.}$$

- Defines a min-cost graph with $k$ arc-disjoint paths between any two players and to/from any player and the depot.
Extension to Network Design Games

- Undirected survivable network design game:
  - Player $i \in N$ has connectivity type $r_i \in \mathbb{Z}_+$.  
  - Must have $r_i$ edge-disjoint 0-$i$ paths.  
  - Must have $r_{ij} = \min\{r_i, r_j\}$ edge-disjoint $i$-$j$ paths.  
  - Some nodes $D \subseteq N \cup 0$ must be parsimonious.

$$C_{r,D}(S) = \min \ cx$$

s.t. $x(\delta_S(i)) = r_i, \quad i \in D \cap S \cup 0$

$$x(\delta_S(U)) \geq \max_{e \in \delta_S(U)} r_e, \quad \emptyset \neq U \subset S \cup 0$$

$$x \geq 0 \text{ integer,}$$

where $r_0 = \max_N r_i$ and $r_{0i} = r_i$ (Goemans/Bertsimas 93).

- Includes MST games, Steiner tree games, undirected edge-connectivity games, ...
Theorem
For both games given above:

1. The games defined by their LP relaxations can be formulated as linear production games.

2. Consequently, these games have allocations in their $\alpha$-core, where the budget balance guarantee $\alpha$ is the integrality gap of these formulations.
We Can Do Better

A 1-2 TSP instance

\[ \begin{align*}
\lambda_1^+ &= 1 \\
\lambda_1^- &= -1/2 \\
\lambda_2^+ &= 2 \\
\lambda_2^- &= -1/2 \\
\lambda_3^+ &= 3/2 \\
\lambda_3^- &= -1 \\
\mu_{23} &= 1/2 \\
\mu_{123} &= 3/2
\end{align*} \]

\[ \begin{align*}
\chi_1 &= 1 \\
\chi_2 &= 9/4 \\
\chi_3 &= 5/4 \\
C(N) &= 5 \\
C_{LP}(N) &= 9/2 \\
\alpha &= 90\% \text{ ("uniform")}
\end{align*} \]

But...
We Can Do Better

A 1-2 TSP instance

\[ \tilde{\chi}_1 = 1 \quad \tilde{\chi}_2 = 2 \quad \tilde{\chi}_3 = 2 \]

- This allocation is efficient and stable.
- Even worse: \( \sum_S \tilde{\chi}_i \leq C_{\text{LP}}(S) \) for \( S \subset N \).
  \( \Rightarrow \) Stable in the LP game!
Conclusions

Can we do this efficiently in general?

\[
\begin{align*}
\max & \quad \alpha \\
\text{s.t.} & \quad \alpha C(N) \leq \chi(N) \leq C(N) \\
& \quad \chi(S) \leq C_{\text{LP}}(S), \quad \emptyset \neq S \subset N
\end{align*}
\]

- Separation problem is “arc-fractional” prize-collecting TSP.

- Complexity unclear.
  - Set function neither sub- nor supermodular.
Conclusions

- Allocation is “fair” and computationally attractive.
  - Empirically, $C_{\text{LP}}(N)$ is very close to $C(N)$.

- Future work: Consider more complex replenishment situations, e.g. inventory routing.

- Contact:
  atoriello@isye.gatech.edu
  www.isye.gatech.edu/~atoriello3/