JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 81, No. 1, APRIL 1994

TECHNICAL NOTE

Directionally Nondifferentiable Metric Projection

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Communicated by E. Polak

Abstract. A closed convex set in $\mathbb{R}^2$ is constructed such that the associated metric projection onto that set is not everywhere directionally differentiable.

Key Words. Metric projection, orthogonal projection, directional differentiability.

Let S be a closed convex subset of $\mathbb{R}^n$, equipped with the Euclidean norm $\|\cdot\|$, and let $P_S(x)$ be the associated metric projection onto S. That is,

$$P_S(x) := \operatorname{argmin}\{\|x-y\| : y \in S\}.$$

It is known (e.g., Ref. 1) that if $x_0 \in S$, then P_S is directionally differentiable at x_0 and the corresponding directional derivatives

$$P'_S(x_0, d) = \lim_{t \to 0+} \frac{P_S(x_0+td) - P_S(x_0)}{t}$$

are given by the metric projection of d onto the tangent cone to S at x_0.

In the case $x_0 \notin S$ directional differentiability of P_S at x_0 is guaranteed only under certain conditions on smoothness of the boundary of S (see Ref. 2 and references therein). Kruskal (Ref. 3) constructed an example of a convex set such that the corresponding metric projection is not everywhere directionally differentiable. Kruskal's example is in $\mathbb{R}^3$ and is quite complicated. In this note we construct a family of sets in $\mathbb{R}^2$ whose metric projections fail to be directionally differentiable.

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Consider a sequence $\alpha_1, \alpha_2, \ldots,$ of positive numbers defined by $\alpha_1 = \pi/2$ and $\alpha_{n+1} = c\alpha_n$ for some constant $0 < c < 1$. Note that the sequence α_n is monotonically decreasing and tends to zero. Let $S \subset \mathbb{R}^2$ be the convex hull of the points $(\cos \alpha_n, \sin \alpha_n)$, $n = 1, 2, \ldots,$ and the points $(0, 0)$ and $(1, 0)$. We show now that P_S is not directionally differentiable at the point $x_0 = (2, 0)$ in the direction $d = (0, 1)$.

Consider the following sequences of positive numbers

$$t_n = \sin \alpha_n + (2 - \cos \alpha_n) \tan(\alpha_n/2 + \alpha_{n-1}/2),$$

$$s_n = \sin \alpha_n + (2 - \cos \alpha_n) \tan(\alpha_n/2 + \alpha_{n+1}/2).$$

Then it is not difficult to verify that $P_S(x_0) = (1, 0)$ and that

$$P_S(x_0 + t_n d) = P_S(x_0 + s_n d) = (\cos \alpha_n, \sin \alpha_n), \qquad n = 2, 3, \ldots.$$

Furthermore,

$$\lim_{n \to \infty} \frac{\sin \alpha_n}{t_n} = \frac{2}{3 + c^{-1}}$$

and

$$\lim_{n \to \infty} \frac{\sin \alpha_n}{s_n} = \frac{2}{3 + c}.$$

Consequently,

$$\lim_{n \to \infty} \frac{\pi_2(P_S(x_0 + t_n d)) - \pi_2(P_S(x_0))}{t_n} \neq \lim_{n \to \infty} \frac{\pi_2(P_S(x_0 + s_n d)) - \pi_2(P_S(x_0))}{s_n},$$

where $\pi_2(a)$ denotes the second coordinate of a point $a \in \mathbb{R}^2$, and hence P_S is not directionally differentiable at the point x_0.

References

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