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On Concepts of Directional Differentiability

A. SHAPIRO[1]

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Abstract. Various definitions of directional derivatives in topological vector spaces are compared. Directional derivatives in the sense of Gâteaux, Fréchet, and Hadamard are singled out from the general framework of σ-directional differentiability. It is pointed out that, in the case of finite-dimensional spaces and locally Lipschitz mappings, all these concepts of directional differentiability are equivalent. The chain rule for directional derivatives of a composite mapping is discussed.

Key Words. Directional derivatives, positively homogeneous mapping, locally Lipschitz mapping, chain rule.

1. Introduction

In modern optimization theory, the concept of directional derivative of a function (mapping) is of fundamental importance. It serves as a basis for deriving first-order necessary, and occasionally sufficient, optimality conditions and for designing numerical algorithms. There are numerous monographs where the theory is presented in detail (see, e.g., Refs. 1-9). In recent years much attention has been attracted to investigation of directional differentiability of functions (mappings) appearing in minimax calculus, nonsmooth analysis, sensitivity analysis of parametrized nonlinear programs, and stochastic programming (e.g., Refs. 1-3, Ref. 7, and Refs. 10-15). Several definitions of directional differentiability with varying degrees of requirements have been introduced and applied to various situations. However, the fact that an extensive theory of differentiation already exists in functional analysis is little known. For an excellent survey of crystallization of the involved ideas and extensive bibliographies, the reader is referred to Averbukh and Smolyanov (Refs. 16 and 17) and Nashed (Ref. 18); see also Yamamuro, Ref. 19.

[1] Professor, Department of Mathematics and Applied Mathematics, University of South Africa, Pretoria, Republic of South Africa.

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The main departure here from classical analysis is that the local approximation is carried out by a positively homogeneous mapping, rather than a linear one. This corresponds to one-sided directional derivatives which can be nonlinear and discontinuous even in the case of finite-dimensional spaces. In this paper, we discuss some aspects of directional differentiability which seem to the author to be relevant to optimization theory. In particular, it will be shown that, in the finite-dimensional case, various definitions of directional differentiability reduce to two concepts of directional derivatives in the sense of Gâteaux and Fréchet. In addition, if the considered mapping is locally Lipschitz, then all those definitions are equivalent.

2. Definitions

Let X and Y be topological vector spaces over the field of real numbers (the topologies are assumed to be Hausdorff), and consider a mapping $f: X \to Y$. Then, f is said to be weakly directionally differentiable at a point $x \in X$ if the limit

$$f'_x(h) = \lim_{t \to 0^+} [f(x+th) - f(x)]/t \tag{1}$$

exists for every $h \in X$. We also say that f is directionally differentiable at x in the sense of Gâteaux.

It immediately follows from the definition that, whenever the (one-sided) directional derivative $f'_x(\cdot)$ exists, it is positively homogeneous, i.e.,

$$f'_x(th) = tf'_x(h), \qquad \text{for all } t \geq 0 \text{ and } h \in X.$$

Therefore, the weak directional differentiability can be formulated in the following form. There exists a positively homogeneous mapping $A: X \to Y$ such that

$$f(x+h) - f(x) = A(h) + r(h), \tag{2}$$

where

$$\lim_{t \to 0^+} r(th)/t = 0,$$

for every $h \in X$. It can be easily shown that, whenever such a mapping A exists, it is unique and coincides with the directional derivative f'_x defined in (1).

Now, following Averbukh and Smolyanov (Ref. 16), we introduce a class of directional derivatives corresponding to the topology of uniform convergence on a family of subsets of X.

Definition 2.1. Let σ be a family of subsets of X. We say that f is σ-directionally differentiable at $x \in X$, if there exists a positively homogeneous mapping $A: X \to Y$ such that the equality (2) holds and, for every $S \in \sigma$, $t^{-1}r(th)$ converges to zero as $t \to 0^+$ uniformly with respect to h in S. That is, for every $S \in \sigma$ and every neighborhood V of 0 in Y, there exists $\delta > 0$ such that $h \in S$ and $t \in (0, \delta)$ imply that $t^{-1}r(th) \in V$.

For the historical background of the development of such an approach to differentiability, see Ref. 17 (Section 2). Key references here are de Lamadrid (Ref. 20) and Sebastião e Silva (Ref. 21).

When the family σ consists of all finite subsets of X, the obtained σ-directional derivative coincides with the weak (Gâteaux) directional derivative defined earlier. We single out two additional important cases: the family σ formed by sequentially compact subsets of X; and the family σ of all bounded subsets of X. We refer to the obtained directional derivatives as compact and bounded directional derivatives, respectively.

If X and Y are normed spaces, then bounded directional differentiability means that

$$\lim_{h \to 0} \|f(x+h) - f(x) - A(h)\| / \|h\| = 0; \tag{3}$$

see Ref. 16, Corollary 1.1, and Ref. 19, p. 9. If in addition the mapping A in (3) is linear and continuous, then it is known as the Fréchet derivative of f at x. Therefore, we call a positively homogeneous mapping A satisfying (3) a directional derivative in the sense of Fréchet. A definition equivalent to (3), for real-valued functions with A restricted to the class of positively homogeneous functions representable as a difference of two sublinear functions, was introduced in Ref. 22. For locally Lipschitz mappings in finite-dimensional spaces, an equivalent definition was also given in Ref. 23 (Definition A.1 and Theorem A.2) under the name of Bouligand derivative.

Compact (two-sided) directional derivatives were first studied in detail by Sova (Refs. 24 and 25). He showed that compact differentiability is equivalent to Hadamard differentiability.

Definition 2.2. We say that f is directionally differentiable at x in the sense of Hadamard if, for any mapping $\varphi: \mathbb{R}_+ \to X$ such that $\varphi(0) = x$ and $t^{-1}(\varphi(t) - \varphi(0))$ converges to a vector h as $t \to 0^+$, the limit

$$\lim_{t \to 0^+} [f(\varphi(t)) - f(x)]/t \tag{4}$$

does exist.

The above definition gives the directional derivative along a curve tangential to h. It readily follows that, whenever the limit (4) exists, it is a function $A(h)$ of the vector h only. Of course, the obtained mapping $A(h)$ is positively homogeneous.

Hadamard directional derivative $A(h)=f'_x(h)$ can also be written in the form

$$f'_x(h)=\lim_{n\to\infty}[f(x+t_nh_n)-f(x)]/t_n, \tag{5}$$

where $\{h_n\}\subset X$ and $\{t_n\}\subset\mathbb{R}_+$ are any sequences such that $h_n\to h$ and $t_n\to 0^+$. Moreover, if the topology of X is metrizable, then one can write (5) as

$$f'_x(h)=\lim_{\substack{h'\to h\\ t\to 0^+}}[f(x+th')-f(x)]/t. \tag{6}$$

Directional derivatives in the sense (6) were employed, for example, in Rockafellar (Ref. 12); see also Refs. 8 and 26.

In the following section, we discuss interrelations between Hadamard, compact, and bounded directional derivatives. Notice that Hadamard differentiability is stronger than Gâteaux differentiability even in the case of finite-dimensional spaces; see, e.g., Ref. 16, Example 1.9.

Finally, we mention the following definition of directional differentiability in normed spaces, due to Demyanov and Rubinov (Ref. 27): The directional derivative $f'_x(\cdot)$ in the sense of Gâteaux does exist and, for any $h\in X$ and $\epsilon>0$, there exist positive numbers δ and η such that the inequality

$$\|[f(x+tv)-f(x)]/t-f'_x(v)\|<\epsilon \tag{7}$$

holds for all $t\in(0,\eta)$ and all $v\in B(h,\delta)$, where $B(h,\delta)$ denotes the open ball

$$B(h,\delta)=\{v:\ \|v-h\|<\delta\}.$$

It will be shown in the next section that directional differentiability in the sense of Demyanov and Rubinov is equivalent to compact directional differentiability. Now, we remark only that, if the term $f'_x(v)$ in the left-hand side of (7) is replaced by $f'_x(h)$, then it becomes a definition of the limit (6). Therefore, if in addition the directional derivative $f'_x(\cdot)$ is continuous, then directional differentiability in the sense of Demyanov and Rubinov is equivalent to directional differentiability in the sense of Hadamard.

3. Basic Results

In this section, we discuss some properties of compact, bounded, and Hadamard directional derivatives and their relations to each other. First, we show that Hadamard differentiability implies the following continuity property of the directional derivative. A mapping $g: X\to Y$ is said to be sequentially continuous at a point x if, for every sequence $\{x_n\}$ converging to x, $\{g(x_n)\}$ tends to $g(x)$. In the case where topology of X is metrizable, sequential continuity is equivalent to continuity.

Proposition 3.1. If f is directionally differentiable at x in the Hadamard sense, then the directional derivative f'_x is sequentially continuous on X.

Proof. Consider a sequence $\{h_n\}$ converging to h, and let V be a neighborhood of 0 in Y. Since f is weakly directionally differentiable, for every n there is a positive number t_n such that

$$[f(x+t_nh_n)-f(x)]/t_n - f'_x(h_n) \in V. \tag{8}$$

Of course, the sequence $\{t_n\}$ can be chosen in such a way that $t_n \to 0^+$. Now, Hadamard differentiability implies that

$$[f(x+t_nh_n)-f(x)]/t_n - f'_x(h) \in V, \tag{9}$$

for n sufficiently large. From (8) and (9), we obtain

$$f'_x(h)-f'_x(h_n) \in V - V.$$

Since the neighborhood V is arbitrary, it follows that $\{f'_x(h_n)\}$ tends to $f'_x(h)$. □

The result of Proposition 3.1 shows that, if X is metrizable, then Hadamard directional differentiability implies continuity of the directional derivative f'_x; see Ref. 6, proposition 4.2.6. On the other hand, without additional assumptions, σ-directional differentiability does not imply the continuity of f'_x, even in the finite-dimensional case. For example, if the mapping f is positively homogeneous, then its directional derivative at zero coincides with f and the remainder $r(h)$ is identically zero. Consequently, in this case, f is σ-directionally differentiable at zero for any family σ. Of course, even in finite-dimensional spaces, there exist positively homogeneous functions which are not continuous. Such discontinuous behavior of f'_x is not possible if f is continuous in a neighborhood of x.

Proposition 3.2. If f is compactly directionally differentiable at x and sequentially continuous in a neighborhood of x, then f'_x is sequentially continuous on X.

Proof. Without loss of generality, we can assume that $x=0$ and $f(0)=0$. Consider a sequence $\{h_n\}$ converging to h, and let V be a neighborhood of 0 in Y. Notice that the set $\{h\} \cup \{h_n : n=1,2,\ldots\}$ is sequentially compact. Therefore, we have that, for all n and sufficiently small positive t,

$$f'_x(th_n) - f(th_n) \in tV, \tag{10}$$

$$f(th) - f'_x(th) \in tV. \tag{11}$$

Moreover, for sufficiently small t, the mapping f is sequentially continuous at th. Consequently we can choose a positive t such that, for all n, the inclusions (10) and (11) hold and, for n sufficiently large,

$$f(th_n) - f(th) \in tV. \tag{12}$$

It follows from (10)–(12) that

$$f'_x(h_n) - f'_x(h) = t^{-1}[f'_x(th_n) - f'_x(th)] \in 3V,$$

and hence the proof is complete. □

For two-sided, continuous directional derivatives, the equivalence of compact and Hadamard differentiability was first shown by Sova (Refs. 24 and 25); see also Ref. 17, Theorem 3.2.

Proposition 3.3. Hadamard directional differentiability implies compact directional differentiability. Conversely, if f possesses compact directional derivative f'_x and if f'_x is sequentially continuous on X, then f is directionally differentiable at x in the Hadamard sense.

Proof. Let f be directionally differentiable at x in the Hadamard sense. In order to show that f is compactly directionally differentiable at x, it is enough to show that, for any sequentially compact set S and sequences $\{h_n\} \subset S$ and $\{t_n\} \to 0^+$,

$$\lim_{n\to\infty} [[f(x + t_n h_n) - f(x)]/t_n - f'_x(h_n)] = 0. \tag{13}$$

Moreover, since S is sequentially compact, we can assume that $\{h_n\}$ converges to a vector h. Also, because of the result of Proposition 3.1, the term $f'_x(h_n)$ in (13) can be replaced by $f'_x(h)$. Hadamard differentiability now implies that the obtained limit is zero.

Conversely, suppose that f is compactly directional differentiable at x and that f'_x is sequentially continuous. Consider a sequence $\{h_n\}$ converging to h and $\{t_n\} \to 0^+$. Since $\{h_n\}$ converges to h, the set $\{h\} \cup \{h_n : n = 1, 2, \ldots\}$ is sequentially compact, and hence the limit (13) holds. Because of the sequential continuity of f'_x, the term $f'_x(h_n)$ can be replaced by $f'_x(h)$ and Hadamard directional differentiability follows. □

Now, we show that directional differentiability in the sense of Demyanov and Rubinov (Ref. 27) is equivalent to compact directional differentiability.

Proposition 3.4. In normed spaces, compact directional derivatives and directional derivatives in the sense of Demyanov and Rubinov are equivalent.

Proof. Notice that, in normed spaces, sequential compactness is equivalent to compactness.

Let f be compactly directionally differentiable at x, and suppose that f is not directionally differentiable at x in the sense of Demyanov and Rubinov. This means that there exist $h \in X$ and $\epsilon > 0$ such that, for any $\delta > 0$ and $\eta > 0$,

$$\|[f(x+tv)-f(x)]/t-f'_x(v)\| \geq \epsilon,$$

for some $t \in (0, \eta)$ and $v \in B(h, \delta)$. This implies that there exist sequences $t_n \to 0^+$ and $v_n \to h$ such that

$$\|[f(x+t_n v_n)-f(x)]/t_n - f'_x(v_n)\| \geq \epsilon. \tag{14}$$

Consider the set

$$S = \{v_n: n = 1, 2, \ldots\} \cup \{h\}.$$

Since $\{v_n\}$ converges, the set S is compact. But (14) contradicts the uniform convergence on S. By the argument of contradiction, this completes the proof that compact directional differentiability implies directional differentiability in the sense of Demyanov and Rubinov.

Conversely, suppose that f is directionally differentiable in the sense of Demyanov and Rubinov. Let S be a compact subset of X, and let ϵ be a positive number. Then, for every $h \in S$, there are $\delta > 0$ and $\eta > 0$ such that (7) holds for all $t \in (0, \eta)$ and $v \in B(h, \delta)$. Since S is compact, one can choose a finite cover $B(h_1, \delta_1), \ldots, B(h_k, \delta_k)$ of S by such balls. Consider the corresponding numbers $\eta_1, \ldots, \eta_k$, and let $\bar{\eta}$ be their minimum. Then, (7) holds for all $v \in S$ and all $t \in (0, \bar{\eta})$. Since ϵ is arbitrary, the required uniform convergence follows. □

Now, let us suppose that the spaces X and Y are normed and that the mapping f is locally Lipschitz; i.e., there exists a positive constant K such that

$$\|f(v)-f(u)\| \leq K\|v-u\|,$$

for all v and u in a neighborhood of x. Then, it can be easily shown that, when the weak directional derivative $f'_x(\cdot)$ exists, it is Lipschitz continuous on X; see e.g., Ref. 27, p. 13, and Ref. 23, Theorem A.2(a). Moreover, in this case, directional differentiability in the senses of Gâteaux and Hadamard are equivalent; see Ref. 18, Theorem 1.13; see also Ref. 27, Lemma 3.2, and Ref. 6, p. 259. Indeed,

$$\begin{aligned}
&\|[f(x+tv)-f(x)]/t-f'_x(h)\| \\
&\leq \|[f(x+th)-f(x)]/t-f'_x(h)\|+\|[f(x+tv)-f(x+th)]\|/t \\
&\leq \|[f(x+th)-f(x)]/t-f'_x(h)\|+K\|v-h\|,
\end{aligned}$$

for t sufficiently small; hence, Gâteaux directional differentiability implies Hadamard directional differentiability.

Proposition 3.5. For locally Lipschitz mappings in normed spaces, Hadamard and Gâteaux directional derivatives are equivalent.

When the space X is finite dimensional, the bounded, Fréchet, compact, and Demyanov-Rubinov directional derivatives are all equivalent. If in addition the directional derivative is continuous, then these derivatives are equivalent to the Hadamard directional derivative. For locally Lipschitz mappings and the finite-dimensional space X, all the above directional derivatives coincide.

Finally, we discuss the chain rule for the directional derivatives of a composite mapping. Let X, Y, and Z be topological vector spaces, and let $f: X \to Y$, $g: Y \to Z$ be some mappings. Suppose that f and g are directionally differentiable in a certain sense at points x and $y=f(x)$, respectively. We address the question: Under what conditions is the composite mapping $g \circ f$ directionally differentiable at x, and under what conditions does the chain rule

$$(g \circ f)'_x = g'_y \circ f'_x \tag{15}$$

hold? In this respect, weak (Gâteaux) directional differentiability certainly is not enough. In Ref. 16, Example 1.22, there are examples of mappings $f: \mathbb{R} \to \mathbb{R}^2$ and $g: \mathbb{R}^2 \to \mathbb{R}$ such that f is Fréchet differentiable and g is Gâteaux differentiable, but the composite mapping $g \circ f: \mathbb{R} \to \mathbb{R}$ is not differentiable (not even continuous) at zero.

Consider another example. Let $f: \mathbb{R} \to \mathbb{R}^2$ be given by

$$\begin{aligned}
&f(t) = (t, \varphi(t)), \\
&\varphi(t) = t^2 \sin(1/t), \qquad t \neq 0, \\
&\varphi(0) = 0;
\end{aligned}$$

and let $g:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$g(y_1, y_2) = y_1, \quad y_2 = 0; \qquad g(y_1, y_2) = 0, \quad y_2 \neq 0.$$

The mapping f is Fréchet differentiable at zero and g is positively homogeneous, hence Fréchet directionally differentiable at (0, 0). However, the composite mapping $g \circ f$ is not directionally differentiable at zero. The pathological behavior of the composite mapping in this example is explained by the discontinuity of the directional derivative of the mapping g.

Proposition 3.6. (i) Let f be Hadamard (Gâteaux) directionally differentiable at x, and let g be Hadamard directionally differentiable at $y = f(x)$. Then, the composite mapping $g \circ f$ is Hadamard (Gâteaux) directionally differentiable at x and the chain rule (15) holds.

(ii) If f and g possess bounded directional derivatives f'_x and g'_y, if the directional derivative f'_x is continuous at zero, and if g'_y is continuous on Y, then $g \circ f$ is boundedly directionally differentiable at x and (15) follows.

Proof. Statement (i) follows immediately from the definitions. Our proof of (ii) is patterned after the proof of Theorem 1.6 in Ref. 16. Suppose that the assumptions of (ii) hold. Without loss of generality, we can assume that

$$x = 0, \qquad y = f(0) = 0, \qquad g(0) = 0.$$

First, we observe that, since f'_x is positively homogeneous and continuous at zero, it maps bounded subsets of X into bounded subsets of Y. Consider a bounded subset S of X. We have that

$$f(th) = tf'_x(h) + r_1(th),$$

where $t^{-1}r_1(th)$ tends to zero as $t \to 0^+$ uniformly on S. Then,

$$g(f(th)) = tg'_y(f'_x(h) + t^{-1}r_1(th)) + r_2[t(f'_x(h) + t^{-1}r_1(th))].$$

Since g'_y is continuous, it follows that $g'_y(f'_x(h) + t^{-1}r_1(th))$ tends to $g'_y(f'_x(h))$ as $t \to 0^+$ uniformly on S. Therefore, it remains to show that

$$t^{-1}r_2[t(f'_x(h) + t^{-1}r_1(th))]$$

tends to zero as $t \to 0^+$ uniformly on S. Suppose that this is false. Then, there exist sequences $\{t_n\} \to 0^+$ and $\{h_n\} \subset S$ such that

$$t_n^{-1}r_2[t_n(f'_x(h_n) + t_n^{-1}r_1(t_nh_n))] \tag{16}$$

does not tend to zero as $n \to \infty$. Now, we have the sequence $\{f'_x(h_n)\}$ is bounded and that $\{t_n^{-1}r_1(t_nh_n)\}$ tends to zero, and hence is also bounded. It follows then from bounded directional differentiability of g that the sequence given in (16) tends to zero, a contradiction. □

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