ISyE 8872  Topics in Nonlinear Optimization

Fall 2001

Assignment 2

Issued: August 28, 2001

Due: September 4, 2001

Problem 1
If you need additional assumptions for a result to hold, then state those assumptions, and motivate why the assumptions are needed.

1. Consider \( f, g: \mathcal{X} \to \mathbb{R} \). Show that

   \[
   \inf_{x \in \mathcal{X}} f(x) + \inf_{x \in \mathcal{X}} g(x) \leq \inf_{x \in \mathcal{X}} \{ f(x) + g(x) \}
   \]

   Give an example where strict inequality holds.

2. Consider \( f: \mathcal{X} \to \mathbb{R} \) and \( g: \mathcal{Y} \to \mathbb{R} \). Show that

   \[
   \inf_{x \in \mathcal{X}} f(x) + \inf_{y \in \mathcal{Y}} g(y) = \inf_{x \in \mathcal{X}, y \in \mathcal{Y}} \{ f(x) + g(y) \}
   \]

3. Consider \( f: \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \). Show that

   \[
   \inf_{x \in \mathcal{X}} \{ \inf_{y \in \mathcal{Y}} f(x, y) \} = \inf_{y \in \mathcal{Y}} \{ \inf_{x \in \mathcal{X}} f(x, y) \} = \inf_{x \in \mathcal{X}, y \in \mathcal{Y}} f(x, y)
   \]

4. Consider \( f: \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \). Show that

   \[
   \sup_{x \in \mathcal{X}} \{ \inf_{y \in \mathcal{Y}} f(x, y) \} \leq \inf_{y \in \mathcal{Y}} \{ \sup_{x \in \mathcal{X}} f(x, y) \}
   \]

   Give an example where strict inequality holds.

Problem 2
For each of the following statements, either prove the statement or give a counterexample:

1. Let \( L_1, L_2 \) be two linear subspaces in \( V \). Then \( L_1 \cup L_2 \) is a linear subspace in \( V \).

2. Let \( A_1, A_2 \) be two affine manifolds in \( V \). Then \( A_1 \cup A_2 \) is an affine manifold in \( V \).

3. Let \( C_1, C_2 \) be two convex sets in \( V \). Then \( C_1 \cup C_2 \) is a convex set in \( V \).

4. Let \( K_1, K_2 \) be two cones in \( V \). Then \( K_1 \cup K_2 \) is a cone in \( V \).

5. Let \( \{ L_\alpha \subseteq V : \alpha \in S \} \) be an arbitrary collection of linear subspaces \( L_\alpha \) in \( V \). Then \( \bigcap \{ L_\alpha \subseteq V : \alpha \in S \} \) is a linear subspace in \( V \).

6. Let \( \{ A_\alpha \subseteq V : \alpha \in S \} \) be an arbitrary collection of affine manifolds \( A_\alpha \) in \( V \). Then \( \bigcap \{ A_\alpha \subseteq V : \alpha \in S \} \) is an affine manifold in \( V \).
(7) Let \( \{ C_\alpha \subseteq V : \alpha \in S \} \) be an arbitrary collection of convex sets \( C_\alpha \) in \( V \). Then \( \bigcap \{ C_\alpha \subseteq V : \alpha \in S \} \) is a convex set in \( V \).

(8) Let \( \{ K_\alpha \subseteq V : \alpha \in S \} \) be an arbitrary collection of cones \( K_\alpha \) in \( V \). Then \( \bigcap \{ K_\alpha \subseteq V : \alpha \in S \} \) is a cone in \( V \).

(9) Let \( \{ K_\alpha \subseteq V : \alpha \in S \} \) be an arbitrary collection of convex cones \( K_\alpha \) in \( V \). Then \( \bigcap \{ K_\alpha \subseteq V : \alpha \in S \} \) is a convex cone in \( V \).

(10) A cone \( K \subseteq V \) is convex if and only if \( K + K \subseteq K \).

(11) A set \( K \subseteq V \) is convex if and only if \( K + K \subseteq K \).

(12) A set \( K \subseteq V \) is a linear subspace if and only if \( K \) is a convex cone and \(-K \subseteq K \).

(13) Let \( \{ L_i \subseteq V_i : i \in \{1, \ldots, n\} \} \) be a collection of linear subspaces. Then the Cartesian product \( L_1 \times \cdots \times L_n \) is a linear subspace in \( V_1 \times \cdots \times V_n \).

(14) Let \( \{ A_i \subseteq V_i : i \in \{1, \ldots, n\} \} \) be a collection of affine manifolds. Then the Cartesian product \( A_1 \times \cdots \times A_n \) is an affine manifold in \( V_1 \times \cdots \times V_n \).

(15) Let \( \{ C_i \subseteq V_i : i \in \{1, \ldots, n\} \} \) be a collection of convex sets. Then the Cartesian product \( C_1 \times \cdots \times C_n \) is a convex set in \( V_1 \times \cdots \times V_n \).

(16) Let \( \{ K_i \subseteq V_i : i \in \{1, \ldots, n\} \} \) be a collection of cones. Then the Cartesian product \( K_1 \times \cdots \times K_n \) is a cone in \( V_1 \times \cdots \times V_n \).

(17) Let \( \{ L_i \subseteq V : i \in \{1, \ldots, n\} \} \) be a collection of linear subspaces in \( V \), and \( c_1, \ldots, c_n \in F \). Then the direct sum \( c_1 L_1 + \cdots + c_n L_n \) is a linear subspace in \( V \).

(18) Let \( \{ A_i \subseteq V : i \in \{1, \ldots, n\} \} \) be a collection of affine manifolds in \( V \), and \( c_1, \ldots, c_n \in F \). Then the direct sum \( c_1 A_1 + \cdots + c_n A_n \) is an affine manifold in \( V \).

(19) Let \( \{ C_i \subseteq V : i \in \{1, \ldots, n\} \} \) be a collection of convex sets in \( V \), and \( c_1, \ldots, c_n \in \mathbb{R} \). Then the direct sum \( c_1 C_1 + \cdots + c_n C_n \) is a convex set in \( V \).

(20) Let \( \{ K_i \subseteq V : i \in \{1, \ldots, n\} \} \) be a collection of cones in \( V \), and \( c_1, \ldots, c_n \in \mathbb{R} \). Then the direct sum \( c_1 K_1 + \cdots + c_n K_n \) is a cone in \( V \).

**Problem 3**

Assume that \( L \subseteq V \) is a linear subspace and \( A \subseteq V \) is an affine manifold. Show that:

(1) for any \( x^0 \in V \), the set \( L + x^0 \) is an affine manifold;

(2) for any \( x^0 \in A \), the set \( A - x^0 \) is a linear subspace which does not depend on \( x^0 \).

(3) \( A \) is a linear subspace if and only if \( 0 \in A \).