Problem 1
Consider closed sets $F, G \subset \mathbb{R}^m$, and suppose that $F \cap G = \emptyset$. Show that there exist open sets $A, B \subset \mathbb{R}^m$ such that $F \subset A$, $G \subset B$, and $A \cap B = \emptyset$.

Problem 2
Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^k$, and $F : X \mapsto 2^Y$.

1. Show that if $F$ is open, then $F$ is lower hemi-continuous.

2. The following statement comes from Proposition 11.9(d) in Border (1985): “If $F$ is upper hemi-continuous at $x$ and $F(x)$ is a singleton, then $F$ is lower hemi-continuous at $x$.”

   (a) Give a counterexample to the statement above.

   (b) Correct the statement, and prove the correctness of your statement.

3. Show that if $F^-(\{y\})$ is open for all $y \in Y$, then $F$ is lower hemi-continuous.

Problem 3
Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^k$, and $F : X \mapsto 2^Y$. Show that $F$ is lower hemi-continuous at $x$ if and only if for all sequences $\{x_k\}_k \subset X$ such that $x_k \rightarrow x$ as $k \rightarrow \infty$, and all $y \in F(x)$, there is a sequence $\{y_k\}_k \subset Y$ such that $y_k \in F(x_k)$ for all $k$ and $y_k \rightarrow y$ as $k \rightarrow \infty$.

Problem 4
Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^k$. Show the following:

1. Consider $F : X \mapsto 2^{\mathbb{R}^m}$. If $F$ is upper hemi-continuous and closed valued, then the set $\{x \in X : x \in F(x)\}$ of fixed points of $F$ is a (possibly empty) closed subset of $X$.

2. Consider $F, G : X \mapsto 2^Y$. If $F$ and $G$ are upper hemi-continuous and closed valued, then the set $\{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is a (possibly empty) closed subset of $X$. 
3. Consider $F : X \mapsto 2^Y$. If $F$ is upper hemi-continuous, then the set $\{ x \in X : F(x) \neq \emptyset \}$ is a closed subset of $X$.

4. Consider $F : X \mapsto 2^Y$. If $F$ is lower hemi-continuous, then the set $\{ x \in X : F(x) \neq \emptyset \}$ is an open subset of $X$.

Problem 5
Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^k$, and $F : X \mapsto 2^Y$. Let $\bar{F} : X \mapsto 2^Y$ be defined by

$$\bar{F}(x) \equiv \text{closure (in $Y$) of } F(x)$$

Show the following:

1. $F$ is lower hemi-continuous at $x$ if and only if $\bar{F}$ is lower hemi-continuous at $x$.

2. If $F$ is upper hemi-continuous at $x$, then $\bar{F}$ is upper hemi-continuous at $x$.

3. Show by example that it is not always true that if $\bar{F}$ is upper hemi-continuous at $x$, then $F$ is upper hemi-continuous at $x$.

Problem 6
Let $G : X \mapsto 2^Y$ and $F : Y \mapsto 2^Z$. Let $F \circ G : X \mapsto 2^Z$ be defined by

$$(F \circ G)(x) \equiv \bigcup_{y \in G(x)} F(y)$$

Show the following:

1. If $F$ and $G$ are upper hemi-continuous, then $F \circ G$ is upper hemi-continuous.

2. If $F$ and $G$ are lower hemi-continuous, then $F \circ G$ is lower hemi-continuous.

3. Show by example that it may happen that $F$ and $G$ are closed, but $F \circ G$ is not closed.