Problem 1
Consider a function $g(x_1, x_2)$, with decisions $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$.

1. Show that
$$\sup_{x_1 \in \mathcal{X}_1} \sup_{x_2 \in \mathcal{X}_2} g(x_1, x_2) = \sup_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2) = \sup_{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2} g(x_1, x_2)$$

2. Show that
$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2) \leq \inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

3. Give an example of a function $g(x_1, x_2)$ such that
$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2) < \inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

4. Suppose two decision makers compete in the following way. Decision maker 1 chooses $x_1 \in \mathcal{X}_1$ with the objective to maximize $g(x_1, x_2)$, and decision maker 2 chooses $x_2 \in \mathcal{X}_2$ with the objective to minimize $g(x_1, x_2)$. Decision maker 1 chooses first, and decision maker 2 chooses second. Is the outcome given by
$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2)$$

or by
$$\inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

Is it better to choose first or to choose second?

Problem 2
Let \( \{ C_i \subseteq V_i : i \in \{1, \ldots, n\} \} \) be a collection of convex sets in vector spaces $V_i$. Show that the Cartesian product $C_1 \times \cdots \times C_n$ is a convex set in $V_1 \times \cdots \times V_n$. 
Problem 3
Let \( \{ C_i \subseteq V_i : i \in \{1, \ldots, n\} \} \) be a collection of convex sets in vector spaces \( V_i \). A function \( f : C_1 \times \cdots \times C_n \mapsto \mathbb{R} \) is called componentwise convex if, for each \( i \in \{1, \ldots, n\} \), and each \( (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in C_1 \times \cdots \times C_{i-1} \times C_{i+1} \times \cdots \times C_n \), the function \( f(x_1, \ldots, x_{i-1}, ,x_{i+1}, \ldots, x_n) : C_i \mapsto \mathbb{R} \) is convex.

1. Show that if \( f \) is convex then \( f \) is componentwise convex.
2. Give an example of a componentwise convex function \( f \) that is not convex.

Problem 4
A function \( f : C \mapsto \mathbb{R} \) is called quasiconvex if all its sublevel sets are convex, that is, if the sets \( \{ x \in C : f(x) \leq a \} \) are convex for all \( a \in \mathbb{R} \).

1. Show that if \( f \) is convex then \( f \) is quasiconvex.
2. Give an example of a quasiconvex function \( f \) that is not convex.

Problem 5
Let \( f : C \mapsto \mathbb{R} \) be a strictly convex function on a convex subset \( C \) of a vector space.

1. Show that \( f \) has at most one minimum on \( C \).
2. Give an example of a strictly convex function \( f \) on a convex set \( C \) such that \( f \) does not have a minimum on \( C \).