ISyE 6761 Stochastic Processes I

Fall 2008

Assignment 7

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Problem 1

Consider a point process on $[0, \infty)$ with points $0 = T_0 \leq T_1 \leq T_2 \leq \cdots$, and $N(t) := \max\{n \geq 0 : T_n \leq t\}$. Prove or disprove with a counterexample:

1. $N(t) < n$ if and only if $T_n > t$.
   Answer: True.

$$\begin{aligned} N(t) < n &\Leftrightarrow \max\{i \geq 0 : T_i \leq t\} < n \\ &\Rightarrow T_n > t \end{aligned}$$

   and

$$\begin{aligned} T_n > t &\Rightarrow T_i \geq T_n > t \ \forall\, i \geq n \\ &\Rightarrow \max\{i \geq 0 : T_i \leq t\} < n \\ &\Leftrightarrow N(t) < n \end{aligned}$$

2. $N(t) \leq n$ if and only if $T_n \geq t$.
   Answer: False. Suppose $T_n < t < T_{n+1}$. Then $N(t) = n$.

3. $N(t) > n$ if and only if $T_n < t$.
   Answer: False. Suppose $t = T_n = T_{n+1}$. Then $N(t) \geq n + 1 > n$.

Problem 2

Give an example of a discrete distribution that is non-arithmetic (non-lattice).

Answer: Example 1: Let $p_n = P[\xi = 1/n] = 1/2^n$ for $n = 1, 2, \ldots$. Note that $\sum_{n=1}^{\infty} p_n = 1$. Then $\{p_n\}$ is a discrete distribution (it has countable support), but there is no $d > 0$ such that $P[\xi = x] > 0$ only if x is an integer multiple of d, because for any $d > 0$, $0 < 1/n < d$ for n sufficiently large.

Example 2: Let $P[\xi = 1] = P[\xi = \sqrt{2}] = 1/2$. Then ξ has a discrete distribution. Suppose the distribution is arithmetic, that is, there is $h > 0$ such that $1 = m_1 h$ and $\sqrt{2} = m_2 h$ for integers m_1, m_2. Then $h = 1/m_1$, thus $\sqrt{2} = m_2/m_1$. But that would imply that $\sqrt{2}$ is a rational number, which it is not. Thus the distribution is non-arithmetic.

Problem 3

Consider a pure renewal process with i.i.d. inter-renewal times $\xi_1, \xi_2, \ldots$, with probability distribution function F with $F(0) < 1$ and $\mathbb{E}[\xi_1] < \infty$. Let $T_0 := 0$ and $T_n := \sum_{i=1}^n \xi_i$. Let $N(t) := \max\{n \geq 0 : T_n \leq t\}$. Define $m_k(t) := \mathbb{E}[N(t)^k]$. Use a renewal argument to show that $m_k(t)$ satisfies the renewal equation

$$m_k(t) \quad = \quad z_k(t) + \int_0^t m_k(t-\tau)\, F(d\tau)$$

for all $k = 1, 2, \ldots$. Be sure to specify $z_k(t)$.

Answer: By conditioning on the first inter-renewal time ξ_1, it follows that

$$\begin{aligned}
m_k(t) &:= \mathbb{E}[N(t)^k] \quad = \quad \mathbb{E}(\mathbb{E}[N(t)^k|\xi_1]) \\
&= \int_{[0,t]} \mathbb{E}[N(t)^k|\xi_1 = \tau]\, F(d\tau) + \int_{(t,\infty]} \mathbb{E}[N(t)^k|\xi_1 = \tau]\, F(d\tau) \\
&= \int_{[0,t]} \mathbb{E}[(1 + N(t-\tau))^k]\, F(d\tau) \\
&= \int_{[0,t]} \mathbb{E}\left[\sum_{i=0}^{k} \binom{k}{i} N(t-\tau)^i\right] F(d\tau) \\
&= \sum_{i=0}^{k-1} \int_{[0,t]} \mathbb{E}\left[\binom{k}{i} N(t-\tau)^i\right] F(d\tau) + \int_{[0,t]} \mathbb{E}\left[N(t-\tau)^k\right] F(d\tau) \\
&= z_k(t) + \int_{[0,t]} m_k(t-\tau)\, F(d\tau)
\end{aligned}$$

where $z_k(t) := \sum_{i=0}^{k-1} \int_{[0,t]} \binom{k}{i} m_i(t-\tau)\, F(d\tau)$.

Problem 4

1. Consider a delayed renewal process with proper inter-renewal time distribution functions G for the first inter-renewal time and F for the remaining inter-renewal times. Let $N(t) := \max\{n \geq 0 : T_n \leq t\}$, and let $A(t) := t - T_{N(t)}$ denote the age at time t. Show that if $t[1 - G(t)] \to 0$ as $t \to \infty$ and F is non-arithmetic (non-lattice) with $F(0) < 1$ and $\int_0^\infty x^2 F(dx) < \infty$, then
$$E[A(t)] \quad \to \quad \frac{\int_0^\infty x^2\, F(dx)}{2\int_0^\infty x\, F(dx)}$$
as $t \to \infty$. (Carefully establish direct Riemann integrability where needed.)
Answer: Note that for any $u \in [0, t]$, $\mathbb{E}[A(t)|T_{N(t)} = u] = t - u$. Thus it is convenient to condition on $T_{N(t)}$. Let $U(t) := \sum_{n=0}^\infty F^{n*}(t)$, and $V(t) := \mathbb{E}[N(t)] = \sum_{n=1}^\infty G * F^{(n-1)*}(t) = G * U(t)$. Then, for any $u \in [0, t]$,
$$F_{T_{N(t)}}(u) \quad = \quad \sum_{n=0}^{\infty} \mathbb{P}[T_n \leq u, T_{n+1} > t]$$

$$
\begin{aligned}
&= 1 - G(t) + \sum_{n=1}^{\infty} \mathbb{P}[T_n \le u, T_{n+1} > t] \\
&= 1 - G(t) + \sum_{n=1}^{\infty} \mathbb{E}(\mathbb{P}[T_n \le u, T_{n+1} > t | T_n]) \\
&= 1 - G(t) + \sum_{n=1}^{\infty} \int_{[0,\infty]} \mathbb{P}[T_n \le u, T_{n+1} > t | T_n = y]\, F_{T_n}(dy) \\
&= 1 - G(t) + \sum_{n=1}^{\infty} \int_{[0,u]} \mathbb{P}[T_{n+1} > t | T_n = y]\, G * F^{(n-1)*}(dy) \\
&= 1 - G(t) + \sum_{n=1}^{\infty} \int_{[0,u]} [1 - F(t-y)]\, G * F^{(n-1)*}(dy) \\
&= 1 - G(t) + \int_{[0,u]} [1 - F(t-y)] \sum_{n=1}^{\infty} G * F^{(n-1)*}(dy) \\
&= 1 - G(t) + \int_{[0,u]} [1 - F(t-y)]\, V(dy)
\end{aligned}
$$

The interchange of the integral and the summation can be justified using the fact that all terms are nonnegative. Thus, for all $u \in (0,t]$, $F_{T_{N(t)}}(du) = [1 - F(t-u)]\, V(du)$. Thus, using the key renewal theorem, it follows that

$$
\begin{aligned}
\mathbb{E}[A(t)] &= \mathbb{E}(\mathbb{E}[A(t)|T_{N(t)}]) \\
&= \int_{[0,t]} (t-u)\, F_{T_{N(t)}}(du) \\
&= t[1 - G(t)] + \int_{[0,t]} (t-u)[1 - F(t-u)]\, V(du) \\
&\to \frac{1}{\mathbb{E}[\xi_2]} \int_0^{\infty} u[1 - F(u)]\, du \\
&= \frac{\int_0^{\infty} u \int_{(u,\infty)} F(dx)\, du}{\int_0^{\infty} x\, F(dx)} \\
&= \frac{\int_{(0,\infty)} \int_0^x u\, du\, F(dx)}{\int_0^{\infty} x\, F(dx)} \\
&= \frac{\int_{(0,\infty)} \frac{x^2}{2}\, F(dx)}{\int_0^{\infty} x\, F(dx)}
\end{aligned}
$$

The interchange of integrals is justified by Fubini's theorem because all terms are nonnegative. (Note that it is not correct that $\mathbb{P}[T_{N(t)} = 0] = \mathbb{P}[\xi_1 > t] = 1 - G(t)$, but rather $\mathbb{P}[T_{N(t)} = 0] = \mathbb{P}[\xi_1 > t] + \sum_{n=1}^{\infty} \mathbb{P}[\xi_1 = 0, \cdots, \xi_n = 0, \xi_{n+1} > t] = 1 - G(t) + \sum_{n=1}^{\infty} G(0)F(0)^{n-1}[1 - F(t)] = 1 - G(t) + \sum_{n=1}^{\infty} \int_{[0,0]} [1 - F(t-y)]\, G * F^{(n-1)*}(dy) = 1 - G(t) + \int_{[0,0]} [1 - F(t-y)] \sum_{n=1}^{\infty} G * F^{(n-1)*}(dy) = 1 - G(t) + \int_{[0,0]} [1 - F(t-y)]\, V(dy)$.) To justify use of the key renewal theorem, we have to verify that $h(u) := u[1 - F(u)]$ is directly Riemann integrable. First, note that for any interval $[0,a]$, for all $u \in [0,a]$, $0 \le h(u) \le a$, that is, h is bounded on each interval $[0,a]$. In addition, the set of discontinuities of F is countable, and thus h is a.e. continuous. It follows that h is Riemann integrable on each interval $[0,a]$. Next we show that $\sum_{k=1}^{\infty} \sup_{u\in[k-1,k)} h(u) < \infty$. Note that for any k and any $u, y \in [k-1,k)$, it holds that $h(u) := u[1-F(u)] \leq (y+1)[1-F(y-1)]$, and thus

$$\begin{aligned}
\sup_{u\in[k-1,k)} h(u) &\leq (y+1)[1-F(y-1)] \\
\Rightarrow \quad \sup_{u\in[k-1,k)} h(u) = \int_{k-1}^{k} \sup_{u\in[k-1,k)} h(u)\,dy &\leq \int_{k-1}^{k} (y+1)[1-F(y-1)]\,dy \\
\Rightarrow \quad \sum_{k=1}^{\infty} \sup_{u\in[k-1,k)} h(u) &\leq \sum_{k=1}^{\infty} \int_{k-1}^{k} (y+1)[1-F(y-1)]\,dy \\
&= \int_0^{\infty} (y+1)[1-F(y-1)]\,dy \\
&= \int_0^{\infty} (y+1) \int_{(y-1,\infty)} F(dx)\,dy \\
&= \int_{(-1,\infty)} \int_0^{x+1} (y+1)\,dy\,F(dx) \\
&= \int_{[0,\infty)} \left[\frac{(x+1)^2}{2} + x + 1\right] F(dx) \\
&= \int_{[0,\infty)} \frac{x^2+4x+3}{2} F(dx) \quad < \quad \infty
\end{aligned}$$

Therefore h is directly Riemann integrable.

2. Show that if G has a finite mean, then $t[1-G(t)] \to 0$ as $t \to \infty$. (Hint: You may use the dominated convergence theorem.)
Answer: Let $\xi \sim G$. For any $t \in (0,\infty)$, let

$$\xi_t(\omega) := \begin{cases} \xi(\omega) & \text{if } \xi(\omega) > t \\ 0 & \text{otherwise} \end{cases}$$

Note that, for each ω, $\xi_t(\omega) = 0$ for all $t \geq \xi(\omega)$, that is, for each ω such that $\xi(\omega) < \infty$, which happens with probability one, $\xi_t(\omega) \to 0$ as $t \to \infty$. Using the fact that, by construction, $\xi_t \leq \xi$ for all t, the assumption that $\mathbb{E}[\xi] < \infty$, and the dominated convergence theorem, it follows that

$$\lim_{t\to\infty} \mathbb{E}[\xi_t] = \mathbb{E}\left[\lim_{t\to\infty} \xi_t\right] = 0$$

Note that for each $t \in (0,\infty)$,

$$\mathbb{E}[\xi_t] = \int_t^{\infty} sG(ds) \geq \int_t^{\infty} tG(ds) = t[1-G(t)] \geq 0$$

Thus $t[1-G(t)] \to 0$ as $t \to \infty$.

Problem 5

Inspection Paradox: Consider a pure renewal process with inter-renewal times $\xi_1, \xi_2, \ldots$, with inter-renewal time distribution function F with $F(0) < 1$ and $\mathbb{E}[\xi_1] < \infty$, points $T_0 := 0$, $T_n := \sum_{i=1}^{n} \xi_i$, and $N(t) := \max\{n \geq 0 : T_n \leq t\}$.

1. Show that for all t, $\xi_{N(t)+1}$ is stochastically greater than or equal to ξ_1, that is

$$\mathbb{P}[\xi_{N(t)+1} > x] \quad \geq \quad 1 - F(x)$$

for all t, x.

Answer: The result clearly holds if $t < 0$ or $x < 0$. Next suppose that $t, x \geq 0$.

$$\begin{aligned}
\mathbb{P}[\xi_{N(t)+1} > x] &= \sum_{n=0}^{\infty} \mathbb{P}[\xi_{N(t)+1} > x, N(t) = n] \\
&= \sum_{n=0}^{\infty} \mathbb{E}(\mathbb{P}[\xi_{n+1} > x, N(t) = n | T_n]) \\
&= \sum_{n=0}^{\infty} \int_{[0,\infty]} \mathbb{P}[\xi_{n+1} > x, N(t) = n | T_n = y]\, F^{n*}(dy) \\
&= \sum_{n=0}^{\infty} \int_{[0,t]} \mathbb{P}[\xi_{n+1} > x, \xi_{n+1} > t - y | T_n = y]\, F^{n*}(dy) \\
&= \int_{[0,t]} \mathbb{P}[\xi_{n+1} > \max\{x, t-y\}] \sum_{n=0}^{\infty} F^{n*}(dy) \\
&= \int_{[0,t]} [1 - F(\max\{x, t-y\})]\, U(dy)
\end{aligned}$$

Case 1: Suppose $t \leq x$. Then

$$\begin{aligned}
\mathbb{P}[\xi_{N(t)+1} > x] &= \int_{[0,t]} [1 - F(x)]\, U(dy) \\
&= [1 - F(x)]U(t) \quad \geq \quad [1 - F(x)]U(0) \quad \geq \quad 1 - F(x)
\end{aligned}$$

Case 2.1: Suppose $t > x$ and $U(t) - U(t-x) \geq 1$. Then

$$\begin{aligned}
\mathbb{P}[\xi_{N(t)+1} > x] &= \int_{[0,t-x]} [1 - F(t-y)]\, U(dy) + \int_{(t-x,t]} [1 - F(x)]\, U(dy) \\
&\geq [1 - F(x)][U(t) - U(t-x)] \quad \geq \quad 1 - F(x)
\end{aligned}$$

Case 2.2: Suppose $t > x$ and $U(t) - U(t-x) < 1$. Then

$$\begin{aligned}
\mathbb{P}[\xi_{N(t)+1} > x] &= \int_{[0,t-x]} [1 - F(t-y)]\, U(dy) + \int_{(t-x,t]} [1 - F(x)]\, U(dy) \\
&= U(t-x) - \int_{[0,t]} F(t-y)\, U(dy) + \int_{(t-x,t]} F(t-y)\, U(dy) \\
&\quad + \int_{(t-x,t]} [1 - F(x)]\, U(dy) \\
&= U(t-x) - F * U(t) - \int_{(t-x,t]} [F(x) - F(t-y)]\, U(dy) + U(t) - U(t-x) \\
&= U(t) - \sum_{n=1}^{\infty} F^{n*}(t) - \int_{(t-x,t]} [F(x) - F(t-y)]\, U(dy) \\
&= F^{0*}(t) - \int_{(t-x,t]} [F(x) - F(t-y)]\, U(dy)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \int_{(t-x,t]} [F(x) - F(t-y)]\, U(dy) \\
&\geq 1 - \int_{(t-x,t]} F(x)\, U(dy) \\
&= 1 - F(x)[U(t) - U(t-x)] \quad \geq \quad 1 - F(x)
\end{aligned}
$$

2. Show that

$$
\mathbb{E}[\xi_{N(t)+1}] \quad \geq \quad \mathbb{E}[\xi_1]
$$

for all t.
Answer:

$$
\begin{aligned}
\mathbb{E}[\xi_{N(t)+1}] &= \int_0^\infty \mathbb{P}[\xi_{N(t)+1} > x]\, dx \\
&\geq \int_0^\infty [1 - F(x)]\, dx \quad = \quad \mathbb{E}[\xi_1]
\end{aligned}
$$

3. Calculate $\mathbb{P}[\xi_{N(t)+1} > x]$ for the case with $F(x) := 1 - e^{-\lambda x}$, $x \geq 0$.
Answer: If $x < 0$, then $\mathbb{P}[\xi_{N(t)+1} > x] = 1$ for all t. If $x \geq 0$ and $t < 0$, then $\mathbb{P}[\xi_{N(t)+1} > x] = \mathbb{P}[\xi_1 > x] = e^{-\lambda x}$.
Next suppose that $t, x \geq 0$. Recall that $U(t) = 1 + \lambda t$.
Case 1: Suppose $t \leq x$. Then

$$
\mathbb{P}[\xi_{N(t)+1} > x] \quad = \quad [1 - F(x)]U(t) \quad = \quad e^{-\lambda x}[1 + \lambda t]
$$

Case 2: Suppose $t > x$. Then

$$
\begin{aligned}
\mathbb{P}[\xi_{N(t)+1} > x] &= \int_{[0,t-x]} [1 - F(t-y)]\, U(dy) + \int_{(t-x,t]} [1 - F(x)]\, U(dy) \\
&= \int_{[0,t-x]} e^{-\lambda(t-y)}\, U(dy) + e^{-\lambda x}[U(t) - U(t-x)] \\
&= e^{-\lambda t} + \int_{(0,t-x]} e^{-\lambda(t-y)} \lambda\, dy + e^{-\lambda x} \lambda x \\
&= e^{-\lambda t} + \left[e^{-\lambda x} - e^{-\lambda t}\right] + e^{-\lambda x} \lambda x \\
&= e^{-\lambda x}[1 + \lambda x]
\end{aligned}
$$

In summary, for all $t, x \geq 0$,

$$
\mathbb{P}[\xi_{N(t)+1} > x] \quad = \quad e^{-\lambda x}[1 + \lambda \min\{t, x\}] \quad > \quad e^{-\lambda x} \quad = \quad 1 - F(x) \quad \forall\, t, x > 0
$$

4. Show by example that it may happen that

$$
\mathbb{E}[\xi_{N(t)+1}] \quad > \quad \mathbb{E}[\xi_1]
$$

for all $t > 0$.
Answer: Consider the case with $F(x) := 1 - e^{-\lambda x}$, $x \geq 0$. Using integration by parts, it

follows that

$$\begin{aligned}
\mathbb{E}[\xi_{N(t)+1}] &= \int_0^\infty \mathbb{P}[\xi_{N(t)+1} > x]\, dx \\
&= \int_0^t \mathbb{P}[\xi_{N(t)+1} > x]\, dx + \int_t^\infty \mathbb{P}[\xi_{N(t)+1} > x]\, dx \\
&= \int_0^t e^{-\lambda x}[1+\lambda x]\, dx + \int_t^\infty e^{-\lambda x}[1+\lambda t]\, dx \\
&= \frac{1-e^{-\lambda t}}{\lambda} - te^{-\lambda t} + \int_0^t e^{-\lambda x}\, dx + [1+\lambda t]\frac{e^{-\lambda t}}{\lambda} \\
&= \frac{2-e^{-\lambda t}}{\lambda} \\
&> \frac{1}{\lambda} \; = \; \mathbb{E}[\xi_1] \quad \forall\, t > 0
\end{aligned}$$

Also note that $\lim_{t\to\infty} \mathbb{E}[\xi_{N(t)+1}] = 2/\lambda = 2\mathbb{E}[\xi_1]$, that is, in the long run the inter-renewal interval that contains t is on average twice as long as the average inter-renewal interval.

5. Interpret the results.
   Answer: As discussed in class, the inter-renewal interval that contains t is more likely to be a relatively long interval than a relatively short interval.

Problem 6
Demand for an item in inventory at a warehouse is a Poisson process with rate α, meaning that customers arrive according to a Poisson process, and each customer demands one unit. When the inventory level hits 0, an order of constant size S units is placed with the factory. Delivery time of the entire order from the factory to the warehouse is exponential with mean $1/\mu$. During that time, demand is lost at a cost of c dollars per unit. The inventory holding cost is h dollars per unit per time, based on the *average* amount of inventory held.

1. Check that i.i.d. cycles $\{C_i\}$ can be discerned.
   Answer: Choose the times when the the inventory level hits 0 as renewal times. Note that these times correspond to an arrival of a customer, and that a replenishment order is placed at these times. Thus, at such times both the delivery times start "fresh" and the customer inter-arrival times start "fresh", that is, at these times there are no remnant times, i.e., the age process of both the replenishment time and the customer inter-arrival time is zero at these times. (We do not even have to use the memoryless property of the exponential distribution, and thus the remarks above hold for any i.i.d. customer inter-arrival times and i.i.d. replenishment times, and not just for Poisson processes.) Then the inter-renewal times are the first times after the orders are delivered that S customers arrive. We can use the memoryless property of the exponential distribution to obtain that $\mathbb{E}[C_i] = 1/\mu + S/\alpha$ (for more details, see the next question).

2. Suppose that the inventory level hits 0 at epoch 0. Let $I(t)$ be the inventory level at epoch t and define

$$A \; := \; \int_0^{C_1} I(t)dt$$

where C_1 is the length of the first cycle. Show that

$$E[A] \quad = \quad \frac{S(S+1)}{2\alpha}$$

Answer: Let $\{Y_n\}$ denote the customer inter-arrival times; $\{Y_n\}$ are i.i.d. exponentially distributed, mean $1/\alpha$. Let $\{Z_n\}$ denote the delivery times; $\{Z_n\}$ are i.i.d. exponentially distributed, mean $1/\mu$. Let $B_n := \sum_{i=1}^{n} Y_i$ denote the arrival time of customer n. Let $N_1 := \inf\{n \geq 1 : B_n \geq Z_1$ denote the index of the first customer who arrives after the first replenishment. Next we show that $B_{N_1} - Z_1$ is exponentially distributed with mean $1/\alpha$ (the memoryless property of the exponential distribution). For any $n \in \{1, 2, \ldots\}$, $s, x \geq 0, t \geq s$,

$$\begin{aligned}
& \mathbb{P}[B_{N_1} - Z_1 > x | N_1 = n, B_{N_1-1} = s, Z_1 = t] \\
= \; & \mathbb{P}[B_{n-1} + Y_n - Z_1 > x | N_1 = n, B_{n-1} = s, Z_1 = t] \\
= \; & \mathbb{P}[s + Y_n - t > x | N_1 = n, B_{n-1} = s, Z_1 = t, Y_{N_1} \geq t - s] \\
= \; & \mathbb{P}[Y_n > t - s + x | N_1 = n, B_{N_1-1} = s, Z_1 = t, Y_n \geq t - s] \\
= \; & \frac{e^{-\alpha(t-s+x)}}{e^{-\alpha(t-s)}} \quad = \quad e^{-\alpha x}
\end{aligned}$$

Specifically, $\mathbb{E}[B_{N_1} - Z_1 | N_1, B_{N_1-1}, Z_1] = 1/\alpha$, and thus $\mathbb{E}[B_{N_1} - Z_1] = 1/\alpha$. Also

$$\begin{aligned}
\mathbb{P}[B_{N_1} - Z_1 > x] \quad &= \quad \mathbb{E}(\mathbb{P}[B_{N_1} - Z_1 > x | N_1, B_{N_1-1}, Z_1]) \\
&= \quad \mathbb{E}(e^{-\alpha x}) \quad = \quad e^{-\alpha x}
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}[C_1] \quad &= \quad \mathbb{E}\left[Z_1 + (B_{N_1} - Z_1) + \sum_{i=1}^{S-1} Y_{N_1+i}\right] \\
&= \quad \mathbb{E}[Z_1] + \mathbb{E}\left(\mathbb{E}\left[(B_{N_1} - Z_1) + \sum_{i=1}^{S-1} Y_{N_1+i} \,\middle|\, N_1\right]\right) \\
&= \quad \mathbb{E}[Z_1] + \sum_{n=1}^{\infty} \mathbb{E}\left[(B_{N_1} - Z_1) + \sum_{i=1}^{S-1} Y_{N_1+i} \,\middle|\, N_1 = n\right] \mathbb{P}[N_1 = n] \\
&= \quad \frac{1}{\mu} + \sum_{n=1}^{\infty} \left(\frac{1}{\alpha} + \sum_{i=1}^{S-1} \mathbb{E}\left[Y_{n+i} | \, N_1 = n\right]\right) \mathbb{P}[N_1 = n] \\
&= \quad \frac{1}{\mu} + \sum_{n=1}^{\infty} \left(\frac{1}{\alpha} + (S-1)\frac{1}{\alpha}\right) \mathbb{P}[N_1 = n] \\
&= \quad \frac{1}{\mu} + \frac{S}{\alpha}
\end{aligned}$$

The second to last inequality used the assumption that $\{Y_1, \ldots, Y_n\}$ and $\{Y_{n+1}, Y_{n+2}, \ldots\}$ are independent (we cannot invoke this assumption without a more careful construction while there is a random index N_1). In addition,

$$\begin{aligned}
A \quad &:= \quad \int_0^{C_1} I(t)dt \\
&= \quad Z_1 \times 0 + (B_{N_1} - Z_1) \times S + Y_{N_1+1} \times (S-1) + \cdots + Y_{N_1+S-1} \times 1
\end{aligned}$$

Thus

$$
\begin{aligned}
\mathbb{E}[A] &= \mathbb{E}[Z_1 \times 0 + (B_{N_1} - Z_1) \times S + Y_{N_1+1} \times (S-1) + \cdots + Y_{N_1+S-1} \times 1] \\
&= \mathbb{E}(\mathbb{E}[(B_{N_1} - Z_1) \times S + Y_{N_1+1} \times (S-1) + \cdots + Y_{N_1+S-1} \times 1 | N_1]) \\
&= \sum_{n=1}^{\infty} \mathbb{E}[(B_{N_1} - Z_1) \times S + Y_{N_1+1} \times (S-1) + \cdots + Y_{N_1+S-1} \times 1 | N_1 = n] \mathbb{P}[N_1 = n] \\
&= \sum_{n=1}^{\infty} \left(\frac{S}{\alpha} + \mathbb{E}[Y_{n+1} \times (S-1) + \cdots + Y_{n+S-1} \times 1 | N_1 = n] \right) \mathbb{P}[N_1 = n] \\
&= \sum_{n=1}^{\infty} \left(\frac{S}{\alpha} + \frac{S-1}{\alpha} + \cdots + \frac{1}{\alpha} \right) \mathbb{P}[N_1 = n] \\
&= \frac{S(S+1)}{2\alpha}
\end{aligned}
$$

3. Find the long run cost per unit time of inventory and lost sales.
Answer: Expected inventory cost during a cycle is $\mathbb{E}[hA] = hS(S+1)/(2\alpha)$. Expected lost sales cost during a cycle is

$$
\begin{aligned}
\mathbb{E}[c(N_1 - 1)] &= c\mathbb{E}(\mathbb{E}[N_1 - 1 | Z_1]) \\
&= c \int_0^{\infty} \mathbb{E}[N_1 - 1 | Z_1 = z] \mu e^{-\mu z} \, dz \\
&= c \int_0^{\infty} \alpha z \mu e^{-\mu z} \, dz \\
&= c \frac{\alpha}{\mu}
\end{aligned}
$$

Recall that the expected length of a cycle is $\mathbb{E}[C_1] = 1/\mu + S/\alpha$. Thus, by renewal reward theory, the long run cost per unit time converges to

$$
\frac{\mathbb{E}[hA] + \mathbb{E}[c(N_1 - 1)]}{\mathbb{E}[C_1]} = \frac{h \frac{S(S+1)}{2\alpha} + c \frac{\alpha}{\mu}}{\frac{1}{\mu} + \frac{S}{\alpha}}
$$

as $t \to \infty$.