Problem 1
Consider a Markov chain with state space $\mathcal{S}$. For any subset $\mathcal{S}' \subset \mathcal{S}$, define the closure $\text{cl}(\mathcal{S}')$ to be the smallest closed set that contains $\mathcal{S}'$.

1. Prove that for any $j \in \mathcal{S}$, $\text{cl}\{j\} = \{k \in \mathcal{S} : j \rightarrow k\}$.
   **Answer:** First we show that $\text{cl}\{j\} \subset \{k \in \mathcal{S} : j \rightarrow k\}$. First, show that $\{k \in \mathcal{S} : j \rightarrow k\}$ is a closed set. Consider any $i \in \{k \in \mathcal{S} : j \rightarrow k\}$ and $l \notin \{k \in \mathcal{S} : j \rightarrow k\}$. If $i \rightarrow l$, then $j \rightarrow i$ and $i \rightarrow l$ imply that $j \rightarrow l$, which contradicts $l \notin \{k \in \mathcal{S} : j \rightarrow k\}$. Thus $i \neq l$. That is, $\{k \in \mathcal{S} : j \rightarrow k\}$ is a closed set. Next, note that $j \not\rightarrow j$, and thus $\{j\} \subset \{k \in \mathcal{S} : j \rightarrow k\}$. Since $\text{cl}\{j\}$ is by definition the smallest closed set that contains $\{j\}$, and $\{k \in \mathcal{S} : j \rightarrow k\}$ is a closed set that contains $\{j\}$, it follows that $\text{cl}\{j\} \subset \{k \in \mathcal{S} : j \rightarrow k\}$. Next, consider any $i \in \{k \in \mathcal{S} : j \rightarrow k\}$ and any set $\mathcal{S}'' \subset \mathcal{S}$ that contains $\{j\}$. If $i \notin \mathcal{S}''$, then $\mathcal{S}''$ is not closed. That is, $i$ is in every closed set that contains $\{j\}$. Therefore $i$ is in the smallest closed set that contains $\{j\}$, that is, $i \in \text{cl}\{j\}$. Hence $\text{cl}\{j\} \supset \{k \in \mathcal{S} : j \rightarrow k\}$. Thus we have shown that $\text{cl}\{j\} = \{k \in \mathcal{S} : j \rightarrow k\}$.

2. For the deterministically monotone Markov chain, what is $\text{cl}\{j\}$?
   **Answer:** $\{j, j+1, j+2, \ldots\}$

3. If $j$ is recurrent, show that $\text{cl}\{j\}$ is the communicating class that contains $j$.
   **Answer:** Let $\mathcal{C}(j)$ denote the communicating class that contains $j$. We want to show that $\text{cl}\{j\} = \mathcal{C}(j)$. We have already shown that $\text{cl}\{j\} = \{k \in \mathcal{S} : j \rightarrow k\}$. Thus it is sufficient to show that $\mathcal{C}(j) = \{k \in \mathcal{S} : j \rightarrow k\}$. First we show that $\mathcal{C}(j) \supset \{k \in \mathcal{S} : j \rightarrow k\}$ and then we show that $\mathcal{C}(j) \subset \{k \in \mathcal{S} : j \rightarrow k\}$.

   It was shown in Proposition 2.10.1 that if $j$ is recurrent and $j \rightarrow k$, then $j \leftarrow k$, that is, $j$ and $k$ are in the same (recurrent) communicating class, thus, $k \in \mathcal{C}(j)$. Hence $\mathcal{C}(j) \supset \{k \in \mathcal{S} : j \rightarrow k\}$. Next, consider any $i \in \mathcal{C}(j)$, that is, $j \leftarrow i$. Then $j \rightarrow i$, thus $i \in \{k \in \mathcal{S} : j \rightarrow k\}$. Hence $\mathcal{C}(j) \subset \{k \in \mathcal{S} : j \rightarrow k\}$. Therefore we have shown that $\mathcal{C}(j) = \{k \in \mathcal{S} : j \rightarrow k\}$.
Problem 2
Consider a Markov chain with state space $S$ that is not necessarily finite, and transition matrix $P$. Suppose that $P^2 = P$.

1. Prove that the Markov chain is aperiodic.
   \textbf{Answer:} Consider any state $i$. We want to show that the period $d(i)$ of state $i$ equals 1. It follows from $P^2 = P$ that $P^n = P$ for all $n \geq 1$. If $p_{i,i} = 0$, then $p_{i,i}^{(n)} = (P^n)_{i,i} = P_{i,i} = p_{i,i} = 0$ for all $n \geq 1$, and thus $\{n \geq 1 : p_{i,i}^{(n)} > 0\} = \emptyset$, and hence by definition $d(i) = 1$. Otherwise, if $p_{i,i} > 0$, then $d(i) = 1$.

2. Prove that $p_{i,j} = p_{j,j}$ for all $i, j \in S$ such that $i \leftrightarrow j$.
   \textbf{Answer:} First we show that $p_{i,j} \leq p_{j,j}$ for all $i, j \in S$. This result does not require that $i \leftrightarrow j$. For any $n \geq 1$,
   \begin{align*}
p_{i,j} &= P_{i,j} = (P^n)_{i,j} = p_{i,j}^{(n)} \\
&= \sum_{k=1}^{n} f_{i,j}^{(k)} p_{j,j}^{(n-k)} \\
&= \sum_{k=1}^{n-1} f_{i,j}^{(k)} p_{j,j} + f_{i,j}^{(n)} 1 \\
&\leq p_{j,j} + P_i[\tau_j(1) = n]
\end{align*}

Note that $\sum_{n=1}^{\infty} P_i[\tau_j(1) = n] = P_i[\tau_j(1) < \infty] \leq 1$, and thus $P_i[\tau_j(1) = n] \to 0$ as $n \to \infty$, that is, for any $\varepsilon > 0$, there is $N$ such that $P_i[\tau_j(1) = n] < \varepsilon$ for all $n \geq N$. Thus, for all $n \geq N$,
\begin{align*}
p_{i,j} < p_{j,j} + \varepsilon
\end{align*}

The inequality above holds for all $\varepsilon > 0$, and thus $p_{i,j} \leq p_{j,j}$ for all $i, j \in S$.

Next we show by contradiction that if $i \leftrightarrow j$, then $p_{i,j} \neq p_{j,j}$. Because $i \leftrightarrow j$, there is $n$ such that $p_{j,i}^{(n)} > 0$. But $p_{j,i}^{(n)} = (P^n)_{j,i} = P_{j,i} = p_{j,i}$, and thus $p_{j,i} > 0$. Suppose that $p_{i,j} < p_{j,j}$. Then
\begin{align*}
p_{j,j} &= P_{j,j} = (P^2)_{j,j} = p_{j,j}^{(2)} \\
&= \sum_{k \in S} p_{j,k} p_{k,j} \\
&= p_{j,i} p_{i,j} + \sum_{k \in S \setminus \{i\}} p_{j,k} p_{k,j} \\
&< p_{j,i} p_{j,j} + \sum_{k \in S \setminus \{i\}} p_{j,k} p_{j,j} \\
&= \sum_{k \in S} p_{j,k} p_{j,j} \\
&= p_{j,j}
\end{align*}

which is a contradiction. Thus $p_{i,j} \neq p_{j,j}$, and therefore $p_{i,j} = p_{j,j}$ for all $i, j \in S$ such that $i \leftrightarrow j$. 
3. Write a stationary distribution \( \pi \) in terms of \( P \).

**Answer:** First we show that for any state \( i \) and any transient state \( j \), \( p_{i,j} = 0 \). It follows from Proposition 2.6.3 that \( \sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty \). Thus \( \sum_{n=1}^{\infty} p_{ij} < \infty \). This implies that \( p_{i,j} = 0 \). Next, it follows from \( \sum_{k \in S} p_{i,k} = 1 \) that \( p_{i,k} > 0 \) for some \( k \in S \), and thus there are some recurrent states \( k \in S \). Consider any recurrent state \( k \in S \). Let \( C(k) \) denote the (recurrent) communicating class that contains \( k \). Note that \( \sum_{i \in C(k)} p_{k,i} = 1 \) \((C(k) \text{ is closed})\), and it follows from the previous part that for all \( i, l \in C(k), p_{i,l} = p_{k,l} = p_{k,l} \). Consider the vector \( \pi \) given by the row of \( P \) corresponding to state \( k \). Clearly \( \sum_{i \in S} \pi_i = \sum_{i \in S} p_{k,i} = 1 \). Also,

\[
(\pi P)_l = \sum_{i \in S} \pi_i P_{i,l} = \sum_{i \in S} p_{k,i} P_{i,l} = \sum_{i \in C(k)} p_{k,i} P_{i,l} = p_{k,l} = \pi_l
\]

Therefore \( \pi \) is a stationary distribution of the Markov chain.

**Problem 3**

Show that a simple random walk with state space \( S = \{0, 1, 2, \ldots\} \) and transition probabilities

\[
P_{0,1} = 1, \ P_{n,n+1} = p, \ P_{n,n-1} = q = 1 - p \ \forall \ n \geq 1
\]

is positive recurrent if \( 0 < p < q \).

**Answer:** We show that the Markov chain is irreducible and has a (unique) stationary distribution. Then it follows from Theorem 2.13.2 that the Markov chain is positive recurrent. First note that \( 0 < p < q \) implies that the Markov chain is irreducible. Next, we write down the balance equations:

\[
\begin{align*}
\pi_0 &= q \pi_1 \\
\pi_1 &= \pi_0 + q \pi_2 \\
\pi_2 &= p \pi_1 + q \pi_3 \\
\pi_n &= p \pi_{n-1} + q \pi_{n+1} \quad \forall \ n \geq 2 \\
\Rightarrow \quad \pi_1 &= \frac{1}{q} \pi_0 \\
\pi_2 &= \frac{p}{q^2} \pi_0 \\
\pi_3 &= \frac{p^2}{q^3} \pi_0 \\
\pi_n &= \frac{p^{n-1}}{q^n} \pi_0 \quad \forall \ n \geq 1
\end{align*}
\]
Also

$$\sum_{n=0}^{\infty} \pi_n = 1$$

$$\Rightarrow \quad \pi_0 + \sum_{n=1}^{\infty} \frac{p^{n-1}}{q^n} \pi_0 = 1$$

$$\Rightarrow \quad \pi_0 + \frac{1/q}{1 - p/q} \pi_0 = 1 \quad \text{if } p/q < 1$$

$$\Rightarrow \quad \pi_0 = \frac{1}{1 + \frac{1/q}{1-p/q}} = \frac{q-p}{2q}$$

$$\Rightarrow \quad \pi_n = \frac{p^{n-1}}{q^n} \frac{q-p}{2q} \quad \forall \, n \geq 1$$