Problem 1
Consider a game where a coin is tossed independently again and again. Every time the coin turns up heads, which happens with probability \( p \in (0, 1) \), the player wins a dollar. Whenever the coin turns up tails, the player loses all his earnings to that point. Let \( X_n(\omega) \) denote the player’s accumulated earnings after the \( n \)th toss.

1. Show that \( X : \Omega \to \{0, 1, 2, \ldots\}^\infty \) is a Markov chain, and write down its transition probabilities. \textbf{Answer:} Consider any history \((i_0, i_1, \ldots, i_n)\). Then
\[
P[X_{n+1} = 0|X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = p[\text{coin turns up tails on throw } n + 1] = 1 - p = P[X_{n+1} = 0|X_n = i_n]
\]
(in the case with \( X_{n+1} = 0 \), the transition probability does not even depend on the current state \( i_n \).) Also,
\[
P[X_{n+1} = i_n + 1|X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = p[\text{coin turns up heads on throw } n + 1] = p = P[X_{n+1} = i_n + 1|X_n = i_n]
\]
Also, for all \( j \notin \{0, i_n + 1\}\),
\[
P[X_{n+1} = j|X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = 0 = P[X_{n+1} = j|X_n = i_n]
\]
Transition matrix
\[
P = \begin{bmatrix}
1 - p & p & 0 & 0 & \cdots \\
1 - p & 0 & p & 0 & \cdots \\
1 - p & 0 & 0 & p & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

2. Show that the Markov chain is irreducible. \textbf{Answer:} Consider any two states \( i, j \in \{0, 1, 2, \ldots\} \). We have to show that there is \( n \in \{0, 1, 2, \ldots\} \) such that \( p_{i,j}^{(n)} > 0 \). If \( i = j \), then \( p_{i,j}^{(0)} = 1 > 0 \). If \( i < j \), then \( p_{i,j}^{(j-i)} = p^{j-i} > 0 \). If \( i > j \), then \( p_{i,j}^{(j+1)} = (1 - p)p^j > 0 \).
3. Calculate the expected hitting time \( \mathbb{E}_i[\tau_i(1)] \) for each \( i \). **Answer:** Note that to go from state \( i \) to state \( i \) in one or more steps, the process has to go from state \( i \) to state 0 and then from state 0 to state \( i \). Thus

\[
\mathbb{E}_i[\tau_i(1)] = \mathbb{E}_i[\tau_0(1)] + \mathbb{E}_0[\tau_i(1)]
\]

Starting in state \( i \), \( \tau_0(1) \) is geometrically distributed with mean \( \frac{1}{1 - p} \), that is,

\[
\mathbb{E}_i[\tau_0(1)] = \frac{1}{1 - p}
\]

Starting in state 0, let \( L_n := \tau_0(n) - \tau_0(n - 1), n = 1, 2, \ldots \), denote the length of excursion \( n \) of the process until it reaches state 0 again. Note that \( \{L_n\} \) is an iid sequence, and that the expected length of an excursion until it returns to state 0 is

\[
\mathbb{E}[L_n] = \mathbb{E}_0[\tau_0(1)] = \frac{1}{1 - p}
\]

Let \( N := \inf\{n : L_n > i\} \) denote the number of the first excursion from state 0 to reach state \( i \). Note that \( N \) is a stopping time with respect to \( \{L_n\} \). The probability that an excursion reaches state \( i \) before it returns to state 0 is \( p^i \), and thus \( \mathbb{E}[N] = 1/p^i \). It follows from Wald’s Identity that

\[
\mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] = \mathbb{E}[N] \mathbb{E}[L_n] = \frac{1}{(1 - p)p^i}
\]

Also

\[
\mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] = \mathbb{E}_i[\tau_i(1)] + \mathbb{E}_0[\tau_0(1)]
\]

\[
\Rightarrow \mathbb{E}_i[\tau_i(1)] = \mathbb{E}_i[\tau_0(1)] + \mathbb{E}_0[\tau_i(1)]
\]

\[
= \mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] = \frac{1}{(1 - p)p^i}
\]

Alternative approach to determine \( \mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] \) (implicitly it still uses the fact that \( N \) is a stopping time with respect to \( \{L_n\} \)): Starting in state 0, call an excursion a successful excursion if it reaches state \( i \), which happens with probability \( p^i \), and an unsuccessful excursion otherwise, which happens with probability \( 1 - p^i \). Note that

\[
\mathbb{E}[L_n] = \frac{1}{1 - p}
\]

\[
\Rightarrow \mathbb{P}[\text{successful excursion}]\mathbb{E}[L_n|\text{successful excursion}]
\]

\[
+ \mathbb{P}[\text{unsuccessful excursion}]\mathbb{E}[L_n|\text{unsuccessful excursion}] = \frac{1}{1 - p}
\]

\[
\Rightarrow p^i \left( i + \frac{1}{1 - p} \right) + (1 - p^i) \mathbb{E}[L_n|\text{unsuccessful excursion}] = \frac{1}{1 - p}
\]
\[ \Rightarrow \mathbb{E}[L_n|\text{unsuccessful excursion}] = \frac{\frac{1}{1-p} - p^i (i + \frac{1}{1-p})}{1 - p^i} \]
\[ = \frac{1 - p^i (i(1 - p) + 1)}{(1 - p)(1 - p^i)} \]

Thus, by conditioning on whether the first excursion is a success or not,

\[ \mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] = \mathbb{P}[\text{first excursion successful}] \mathbb{E} \left[ \sum_{n=1}^{N} L_n | \text{first excursion successful} \right] \]
\[ + \mathbb{P}[\text{first excursion unsuccessful}] \mathbb{E} \left[ \sum_{n=1}^{N} L_n | \text{first excursion unsuccessful} \right] \]
\[ = p^i \mathbb{E}[L_n | \text{successful excursion}] \]
\[ + (1 - p^i) \left( \mathbb{E}[L_n | \text{unsuccessful excursion}] + \mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] \right) \]
\[ = p^i \left( i + \frac{1}{1-p} \right) + \frac{1 - p^i (i(1 - p) + 1)}{(1 - p)(1 - p^i)} + \mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] \]
\[ \Rightarrow p^i \mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] = p^i \left( i + \frac{1}{1-p} \right) + \frac{1 - p^i (i(1 - p) + 1)}{1-p} = \frac{1}{1-p} \]
\[ \Rightarrow \mathbb{E} \left[ \sum_{n=1}^{N} L_n \right] = \frac{1}{(1-p)p^i} \]

4. Classify the Markov chain. **Answer:** The Markov chain is irreducible (one class) and positive recurrent.

**Problem 2**
Show that in an irreducible discrete time Markov chain with \( N \) states, it is possible to go from any state to any other state in \( N \) steps or less. **Answer:** Consider any states \( i, j \in S \).

The Markov chain being irreducible implies that there is \( n \) such that \( p_{ij}^{(n)} > 0 \). Note that

\[ p_{ij}^{(n)} := \mathbb{P}[X_n = j | X_0 = i] \]
\[ = \frac{\mathbb{P}[X_0 = i, X_n = j]}{\mathbb{P}[X_0 = i]} \]
\[ = \sum_{k_1,\ldots,k_{n-1}\in S} \frac{\mathbb{P}[X_0 = i, X_1 = k_1, \ldots, X_n = j]}{\mathbb{P}[X_0 = i]} > 0 \]

Thus there is a sequence of states \( i = k_0, k_1, \ldots, k_{n-1}, k_n = j \in S \) such that

\[ \mathbb{P}[X_0 = i, X_1 = k_1, \ldots, X_{n-1} = k_{n-1}, X_n = j] = \mathbb{P}[X_0 = i] p_{k_0,k_1} p_{k_1,k_2} \cdots p_{k_{n-2},k_{n-1}} p_{k_{n-1},k_n} > 0 \]
Suppose any state repeats in this sequence, say $k_l = k_m$ with $l < m$. Then all the states between $k_l$ and $k_m$ can be eliminated from the sequence:

$$P_{k_0, k_1} \cdots P_{k_{l-1}, k_l} P_{k_m, k_{m+1}} \cdots P_{k_{n-1}, k_n} \geq P_{k_0, k_1} P_{k_1, k_2} \cdots P_{k_{n-2}, k_{n-1}} P_{k_{n-1}, k_n} > 0$$

By induction, all duplicated states can be eliminated from the sequence. Because the Markov chain has $N$ states, the resulting sequence has $N$ or fewer states. The Markov chain can go from state $i$ to state $j$ along the resulting sequence of states in $N$ steps or less.

**Problem 3**

For a discrete time Markov chain $X : \Omega \mapsto \mathcal{S}^\infty$, use only basic identities in probability and the Markov property

$$P[X_{n+1} = j|X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = P[X_{n+1} = j|X_n = i_n] = p_{i_n, j}$$

for all histories $(i_0, i_1, \ldots, i_n)$ and all $j$, to prove that

$$P[X_n = j|X_{n_1} = i_1, \ldots, X_{n_k} = i_k] = P[X_n = j|X_{n_k} = i_k]$$

for all states $i_1, \ldots, i_k$ and all $j$, whenever $n_1 < n_2 < \cdots < n_k < n$. **Answer:** First show that

$$P[X_{n+1} = k_1, \ldots, X_{n+m} = k_m|X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i]$$

$$= P[X_{n+1} = k_1, \ldots, X_{n+m} = k_m|X_n = i]$$

for all states $i_0, \ldots, i_{n-1}, i, k_1, \ldots, k_m$.

$$P[X_{n+1} = k_1, \ldots, X_{n+m} = k_m|X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i]$$

$$= \sum_{i_0, \ldots, i_{n-1} \in \mathcal{S}} \frac{P[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1, \ldots, X_{n+m} = k_m]}{P[X_n = i]}$$

$$= \sum_{i_0, \ldots, i_{n-1} \in \mathcal{S}} \frac{P[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i]P[X_{n+1} = k_1|X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i]}{P[X_n = i]}$$

$$\times \cdots \times \frac{P[X_{n+m} = k_m|X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1, \ldots, X_{n+m-1} = k_{m-1}]}{P[X_n = i]}$$

$$= \sum_{i_0, \ldots, i_{n-1} \in \mathcal{S}} \frac{P[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i]}{P[X_n = i]} p_{i_1, k_1} p_{k_1, k_2} \cdots p_{k_{m-1}, k_m}$$

$$= \frac{P[X_n = i]}{P[X_n = i]} p_{i_1, k_1} p_{k_1, k_2} \cdots p_{k_{m-1}, k_m}$$

$$= p_{i, k_1} p_{k_1, k_2} \cdots p_{k_{m-1}, k_m}$$
Next consider the result to be shown.

\[
\mathbb{P}[X_n = j | X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k] = \frac{\mathbb{P}[X_n = j, X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]}{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]}
\]

\[
= \sum_{j_0^{(1)} \cdots j_k^{(n_k-1)}} \mathbb{P}[X_0 = j_0^{(1)}, \ldots, X_{n_k} = i_k] \\
\times \mathbb{P}[X_{n_1} = j_1^{(1)}, \ldots, X_{n_2-1} = j_1^{(n_2-1)}, X_n = j | X_0 = j_0^{(1)}, \ldots, X_{n_k} = i_k] \\
\times \mathbb{P}[X_{n_1} = j_1^{(1)}, \ldots, X_{n_2-1} = j_1^{(n_2-1)}, X_n = j | X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]
\]

\[
= \sum_{j_0^{(1)} \cdots j_k^{(n_k-1)}} \left( \sum_{j_k^{(n_k-1)}} \mathbb{P}[X_{n_k} = i_k, X_{n_k+1} = j_k^{(1)}, \ldots, X_{n-1} = j_k^{(n_k-1)}, X_n = j] \right) \\
\times \mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]
\]

\[
= \left( \sum_{j_0^{(1)} \cdots j_k^{(n_k-1)}} \mathbb{P}[X_0 = j_0^{(1)}, \ldots, X_{n_k} = i_k] \right) \\
\times \left( \sum_{j_k^{(n_k-1)}} \mathbb{P}[X_{n_k} = i_k, X_{n_k+1} = j_k^{(1)}, \ldots, X_{n-1} = j_k^{(n_k-1)}, X_n = j] \right) \\
\times \mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]
\]

(note that the second sum does not depend on the variables in the first sum)

\[
= \mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k] \times \frac{\mathbb{P}[X_{n_k} = i_k, X_n = j]}{\mathbb{P}[X_{n_k} = i_k]} \\
= \mathbb{P}[X_n = j | X_{n_k} = i_k]
\]

**Problem 4**

Suppose that $X : \Omega \mapsto S^\infty$ is a discrete time Markov chain with a countable state space $S$. 
Let \( f : S \mapsto S \) be an arbitrary function. Let \( Y : \Omega \mapsto S^\infty \) be given by \( Y_n(\omega) := f(X_n(\omega)) \). Is \( Y \) a Markov chain? Prove or give a counterexample. If you give a counterexample, also give a sufficient condition on \( f \) for \( Y \) to be a Markov chain. Answer: No, not necessarily. Counterexample: Let \( X \) be a simple random walk on the integers with \( p = q = 1/2 \) and \( X_0 = 0 \). Let \( f(x) = \lfloor x \rfloor := \max\{x, 0\} \). Then

\[
P[f(X_2) = 0, f(X_3) = 0] = P[X_0 = 0, X_1 = -1, X_2 = -2, X_3 = -3] \]
\[
+ P[X_0 = 0, X_1 = -1, X_2 = -2, X_3 = -1] 
+ P[X_0 = 0, X_1 = -1, X_2 = 0, X_3 = -1] 
+ P[X_0 = 0, X_1 = 1, X_2 = 0, X_3 = -1] 
= 4 \left( \frac{1}{2} \right)^3 = \frac{1}{2}
\]

and

\[
P[f(X_2) = 0] = P[X_0 = 0, X_1 = -1, X_2 = -2] 
+ P[X_0 = 0, X_1 = -1, X_2 = 0] 
+ P[X_0 = 0, X_1 = 1, X_2 = 0] 
= 3 \left( \frac{1}{2} \right)^2 = \frac{3}{4}
\]

thus

\[
P[f(X_3) = 0|f(X_2) = 0] = \frac{P[f(X_2) = 0, f(X_3) = 0]}{P[f(X_2) = 0]} = \frac{1/2}{3/4} = \frac{2}{3}
\]

However,

\[
P[f(X_3) = 0|f(X_0) = 0, f(X_1) = 1, f(X_2) = 0] = P[f(X_3) = 0|X_0 = 0, X_1 = 1, X_2 = 0] 
= P[X_3 = -1|X_0 = 0, X_1 = 1, X_2 = 0] 
= P[X_3 = -1|X_2 = 0] = \frac{1}{2}
\]

Thus

\[
P[f(X_3) = 0|f(X_0) = 0, f(X_1) = 1, f(X_2) = 0] \neq P[f(X_3) = 0|f(X_2) = 0]
\]

If \( f \) is a one-to-one function, then \( Y \) is a Markov chain.