Problem 1
Consider a sequence \( \{a_n\} \subset [0, \infty), a \neq 0 \) (not all \( a_n \) are 0), and a sequence \( \{b_n\} \subset \mathbb{R} \). Suppose that
\[
\lim_{n \to \infty} a_n \sum_{k=0}^{n-1} a_k = 0
\]
and
\[
\lim_{n \to \infty} b_n = \bar{b}
\]
Show that
\[
\lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} a_k b_{n-k}}{\sum_{k=0}^{n-1} a_k} = \bar{b}
\]
(This result can be used for later problems.)

Problem 2
Consider a set \( \Lambda \subset \{0, 1, \ldots\} \). Suppose that \( \Lambda \) is closed under addition and that the greatest common divisor of \( \Lambda \) is \( d \). Show that there exists \( N \) such that \( n d \in \Lambda \) for all \( n \geq N \). (This result can also be used for later problems.)

Problem 3
Consider a Markov chain with a state \( i \) with period
\[
d(i) := \text{gcd} \left\{ n \geq 1 : p^{(n)}_{i,i} > 0 \right\}
\]
Show that
\[
d(i) = \text{gcd} \left\{ n \geq 1 : f^{(n)}_{i,i} > 0 \right\}
\]
(Not surprisingly, this result can also be used for later problems.)
Problem 4
Consider a Markov chain with a communicating class $C$ with period $d$. Show that for every $i, j \in C$ there is $r_{i,j} \in \{0, 1, \ldots, d - 1\}$ such that for every $n$ such that $p_{i,j}^{(n)} > 0$, it holds that $n = qd + r_{i,j}$ for some nonnegative integer $q$ ($q$ depends on $n$, but $r_{i,j}$ does not depend on $n$). In addition, show that there exists $N_{i,j}$ such that $p_{i,j}^{(nd+r_{i,j})} > 0$ for all $n \geq N_{i,j}$. (By now you know that this result can also be used for later problems.)

Problem 5
Consider a Markov chain with a recurrent state $i$ with period $d(i)$.

1. Let

$$r_{i,i}^{(n)} := \sum_{k=n+1}^{\infty} f_{i,i}^{(k)}$$

(this $r$ is not the same as the $r$ in the previous problem, but it seems the most natural notation). Show that

$$\sum_{k=0}^{n} r_{i,i}^{(k)} p_{i,i}^{(n-k)} = 1$$

for all $n \geq 0$.

2. Let

$$\lambda := \limsup_{n \to \infty} p_{i,i}^{(nd(i))}$$

and let $\{n_k\} \subset \mathbb{N}$ be a subsequence such that

$$\lim_{k \to \infty} p_{i,i}^{(n_kd(i))} = \lambda$$

(As you know, such a subsequence always exists.) Consider any $t$ such that $f_{i,i}^{(t)} > 0$. Show that

$$\lambda \leq f_{i,i}^{(t)} \liminf_{k \to \infty} p_{i,i}^{(n_kd(i)-t)} + \left(1 - f_{i,i}^{(t)}\right) \lambda$$

Show that

$$\lim_{k \to \infty} p_{i,i}^{(n_kd(i)-t)} = \lambda$$

3. Conclude that

$$\lim_{k \to \infty} p_{i,i}^{(n_kd(i)-\tau)} = \lambda$$

for any $\tau$ of the form $\tau = \sum_{j=1}^{\ell} c_j t_j$ for positive integers $c_j, t_j$ such that $f_{i,i}^{(t_j)} > 0$.

4. Show that there exists $N$ such that for all $n \geq N$,

$$nd(i) = \sum_{j=1}^{\ell} c_j t_j$$

for positive integers $c_j, t_j$ such that $f_{i,i}^{(t_j)} > 0$.
5. Conclude that for all \( n \geq N \),
\[
\lim_{k \to \infty} p_{i,i}^{(n_k-n)d(i)} = \lambda
\]

6. Show that for any \( n_k \geq N \),
\[
\sum_{j=0}^{n_k-N} r_{i,i}^{(jd(i))} p_{i,i}^{(n_k-N-j)d(i)} = 1
\]

7. Show that
\[
\sum_{j=0}^{\infty} r_{i,i}^{(jd(i))} = \frac{\mathbb{E}_i[\tau_1(1)]}{d(i)}
\]

8. Show that, if \( i \) is positive recurrent, then
\[
\lambda \sum_{j=0}^{\infty} r_{i,i}^{(jd(i))} = 1
\]

9. Show that, if \( i \) is null recurrent, then \( \lambda = 0 \).

10. Conclude that
\[
\lambda = \frac{1}{\sum_{j=0}^{\infty} r_{i,i}^{(jd(i))}} = \frac{d(i)}{\mathbb{E}_i[\tau_1(1)]}
\]
   for any recurrent state \( i \).

11. Observe (you don’t have to repeat the details) that the same argument used for \( \lambda := \lim\sup_{n \to \infty} p_{i,i}^{(nd(i))} \) can be used to show that
\[
\liminf_{n \to \infty} p_{i,i}^{(nd(i))} = \frac{d(i)}{\mathbb{E}_i[\tau_1(1)]}
\]

12. Conclude that
\[
\lim_{n \to \infty} p_{i,i}^{(nd(i))} = \frac{d(i)}{\mathbb{E}_i[\tau_1(1)]}
\]

13. Conclude that, if state \( i \) is null recurrent, then
\[
\lim_{n \to \infty} p_{i,i}^{(n)} = 0
\]
   (Note that the superscript in \( p_{i,i}^{(n)} \) is now \( n \) and not \( nd(i) \).)

14. Consider a Markov chain with a communicating class \( \mathcal{C} \) with period \( d \). Show that for any states \( i, j \in \mathcal{C} \),
\[
\lim_{n \to \infty} p_{i,j}^{(nd+r_{i,j})} = \frac{d}{\mathbb{E}_j[\tau_j(1)]}
\]
15. Use the previous result to show that positive recurrence (and thus also null recurrence) is a class property.

**Problem 6**

Consider the following result. Consider a Markov chain with state \( i \) with period \( d(i) \), and suppose that

\[
m^{(2)}_{i,i} := \sum_{k=1}^{\infty} k^2 f_{i,i}^{(k)} < \infty
\]

Then

\[
\lim_{n \to \infty} \left[ \sum_{k=0}^{n} p_{i,i}^{(kd(i))} - \frac{(n + 1)d(i)}{\mathbb{E}_i[\tau_i(1)]} \right] = \frac{m^{(2)}_{i,i} - \mathbb{E}_i[\tau_i(1)]d(i)}{2(\mathbb{E}_i[\tau_i(1)])^2}
\]

Show that this result implies that

\[
\lim_{n \to \infty} p_{i,i}^{(nd(i))} = \frac{d(i)}{\mathbb{E}_i[\tau_i(1)]}
\]

(Clearly, it gives even more information, because it gives a rate of convergence.)