Problem 1
Consider a game where a coin is tossed independently again and again. Every time the coin turns up heads, which happens with probability $p \in (0, 1)$, the player wins a dollar. Whenever the coin turns up tails, the player loses all his earnings to that point. Let $X_n(\omega)$ denote the player’s accumulated earnings after the $n$th toss.

1. Show that $X : \Omega \mapsto \{0, 1, 2, \ldots\}^\infty$ is a Markov chain, and write down its transition probabilities.

2. Show that the Markov chain is irreducible.

3. Calculate the expected hitting time $E_i[\tau_i(1)]$ for each $i$.

4. Classify the Markov chain.

Problem 2
Show that in an irreducible discrete time Markov chain with $N$ states, it is possible to go from any state to any other state in $N$ steps or less.

Problem 3
For a discrete time Markov chain $X : \Omega \mapsto S^\infty$, use only basic identities in probability and the Markov property

\[ \mathbb{P}[X_{n+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = \mathbb{P}[X_{n+1} = j | X_n = i_n] = p_{i_n, j} \]

for all histories $(i_0, i_1, \ldots, i_n)$ and all $j$, to prove that

\[ \mathbb{P}[X_n = j | X_{n_1} = i_1, \ldots, X_{n_k} = i_k] = \mathbb{P}[X_n = j | X_{n_k} = i_k] \]

for all states $i_1, \ldots, i_k$ and all $j$, whenever $n_1 < n_2 < \cdots < n_k < n$.

Problem 4
Suppose that $X : \Omega \mapsto S^\infty$ is a discrete time Markov chain with a countable state space $S$. Let $f : S \mapsto S$ be an arbitrary function. Let $Y : \Omega \mapsto S^\infty$ be given by $Y_n(\omega) := f(X_n(\omega))$. Is $Y$ a Markov chain? Prove or give a counterexample.