ISyE 6664 Stochastic Optimization

Fall 2009

Assignment 1 (Revision)

Issued: August 20, 2009

Due: August 27, 2009

Problem 1

Integration Preserves Convexity: Consider a function $G : \mathcal{X} \times \Omega \mapsto \mathbb{R}$, where $\mathcal{X}$ is a vector space. Suppose $G(x, \omega)$ is convex in x for each $\omega \in \Omega$. Let P be any probability distribution on Ω. Assume that the necessary measurability conditions hold. Show that $g(x) := E_P[G(x, \omega)]$ is convex in x. (If you need additional assumptions for the result to hold, then state those assumptions and motivate why the assumptions are needed.)

Problem 2

Consider a function $G : \mathcal{X} \times \Omega \mapsto \mathbb{R}$, where $\Omega = \mathbb{R}^n$ for some n. Suppose $G(x, \omega)$ is convex in ω for each $x \in \mathcal{X}$. Let P be any probability distribution on Ω. Assume that the necessary measurability conditions hold and that $E_P[\omega]$ is finite. Show that $G(x, E_P[\omega]) \leq E_P[G(x, \omega)]$ for each $x \in \mathcal{X}$. That is, the objective function $G(x, E_P[\omega])$ of the deterministic optimization problem $\min_{x \in \mathcal{X}} G(x, E_P[\omega])$ that replaces ω with its mean is a biased estimate of the objective function $E_P[G(x, \omega)]$ of the stochastic optimization problem $\min_{x \in \mathcal{X}} E_P[G(x, \omega)]$. (If you cannot prove the result for $\Omega = \mathbb{R}^n$, then prove the result for $\Omega = \mathbb{R}$. For bonus points, give a counterexample for the assertion above for a function $G : \mathcal{X} \times \Omega \mapsto \mathbb{R} \cup \{+\infty\}$, where Ω is a vector space.)

Problem 3

Consider a function $g : \mathcal{X}_1 \times \mathcal{X}_2 \mapsto \mathbb{R}$, with decisions $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$.

1. Show that

$$\sup_{x_1 \in \mathcal{X}_1} \sup_{x_2 \in \mathcal{X}_2} g(x_1, x_2) \quad = \quad \sup_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2) \quad = \quad \sup_{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2} g(x_1, x_2)$$

2. Show that

$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2) \quad \leq \quad \inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

3. Give an example of a function $g(x_1, x_2)$ such that

$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2) \quad < \quad \inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

4. Suppose two decision makers compete in the following way. Decision maker 1 chooses $x_1 \in \mathcal{X}_1$ with the objective to maximize $g(x_1, x_2)$, and decision maker 2 chooses $x_2 \in \mathcal{X}_2$ with the objective to minimize $g(x_1, x_2)$. Decision maker 1 chooses first, and decision maker 2 chooses second. Is the outcome given by

$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2)$$

or by

$$\inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

Is it better to choose first or to choose second?

Problem 4

Consider a function $G : \mathcal{X} \times \Omega \mapsto \mathbb{R}$, with decision $x \in \mathcal{X}$, and $\omega \in \Omega$, and probability distribution P on Ω. Show that

$$\sup_{x \in \mathcal{X}} E_P \left[G(x, \omega) \right] \quad \leq \quad E_P \left[\sup_{x \in \mathcal{X}} G(x, \omega) \right]$$