Problem 1
Program a trust region algorithm that uses the dogleg method. Use your program to minimize the Rosenbrock function
\[ f(x_1, x_2) := 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \]
For the second order term, choose \( B_k = \nabla^2 f(x_k) \). Choose \( \eta_1 = 0.1, \eta_2 = 0.2, \eta_3 = 0.9, \beta = 1/2, \gamma = 1, c_1 = 1/2 \). Use \( \| \cdot \|_2 \) and another version with \( \| \cdot \| = \| \cdot \|_Q \), with \( Q = \nabla^2 f(x^*) \). First try the initial point \( x_0 = (1.2, 1.2) \), and then the initial point \( x_0 = (-1.2, 1.0) \). Plot a graph of the distance \( \| x_k - x^* \|_2 \) between the iterate \( x_k \) and the optimal solution \( x^* \) versus iteration index \( k \) for each norm and each initial point. Interpret the results.

Problem 2
Suppose that the objective function \( f : \mathbb{R}^n \mapsto \mathbb{R} \) is bounded below and continuously differentiable. A trust region algorithm is used, with matrices \( B_k \) for the second order term of the model functions satisfying \( \| B_k \| \leq \beta \) for all \( k \). The approximate solutions of the subproblem at each iteration satisfy the two conditions of class. Show that if the iterates remain in a bounded set, then there is a limit point \( x^\infty \) of the sequence \( \{ x_k \} \) such that \( \nabla f(x^\infty) = 0 \).

Problem 3
The Cauchy-Schwartz inequality states that for any \( u, v \in \mathbb{R}^n \), it holds that
\[ |u^T v|^2 \leq (u^T u)(v^T v) \]
with equality only when \( u \) and \( v \) are parallel. Use the Cauchy-Schwartz inequality to show that, for any \( Q \in \mathbb{R}^{n \times n} \) positive definite and for any \( w \in \mathbb{R}^n \), it holds that
\[ |w|^4_2 \leq (w^T Qw)(w^T Q^{-1}w) \]
with equality only when \( w \) and \( Qw \) (and thus also \( Q^{-1}w \)) are parallel.

Problem 4
Show that if \( B \in \mathbb{R}^{n \times n} \) is symmetric, then \( B + \lambda I \) is positive definite for all \( \lambda \) sufficiently large.

Problem 5
Equivalent trust-region methods:
Consider a trust region method with subproblems given by
\[ \min_{p \in \mathbb{R}^n} \left\{ m_1^k(p) := f(x_k) + \nabla f(x_k)^T p + \frac{1}{2} p^T H_k p \right\} \quad \text{subject to} \quad \|p\|_2 \leq \Delta_k \]
Consider another trust region method with subproblems given by
\[ \min_{p \in \mathbb{R}^n} \left\{ m_2^k(p) := f(x_k) + \nabla f(x_k)^T p + \frac{1}{2} p^T (H_k + \lambda_k I)p \right\} \]
(Note that the subproblem above is an unconstrained optimization problem.)
1. Show that the two trust region methods are equivalent, in the sense that for every $\Delta_k > 0$, there is a $\lambda_k \geq 0$ such that $H_k + \lambda_k I$ is positive semidefinite, $\nabla f(x_k)$ is in the range of $H_k + \lambda_k I$ (which automatically holds if $H_k + \lambda_k I$ is positive definite and thus nonsingular), and every optimal solution of the first subproblem is an optimal solution of the second subproblem (in particular, if $H_k + \lambda_k I$ is positive definite then the two subproblems have the same optimal solution); and for every $\lambda_k \geq 0$ and every optimal solution of the second subproblem (which implies that $H_k + \lambda_k I$ is positive semidefinite and $\nabla f(x_k)$ is in the range of $H_k + \lambda_k I$), there is a $\Delta_k > 0$ such that the optimal solution of the second subproblem is also optimal for the first subproblem (again, if $H_k + \lambda_k I$ is positive definite then the two subproblems have the same optimal solution).

2. List some potential advantages/disadvantages of each of the two trust region methods.

**Problem 6**
Consider the following two-dimensional subspace minimization problem (which gives a solution to the subproblem at least as good as that of the dogleg method).

$$\min_{p \in \mathbb{R}^n} \left\{ m(p) := a + b^T p + \frac{1}{2} p^T B p \right\}$$
subject to

\[\|p\|_2 \leq \Delta \]
\[p \in \text{span}[b, B^{-1}b]\]

Suppose that $B$ is positive definite. Describe (in a precise way) a procedure to solve the two-dimensional subspace minimization problem above.