ISyE 4803

Advanced Supply Chain Logistics

Fall 2009

Homework 2 solution

Question 1 (25 points)

Below is a network, with costs indicated on the arcs, and load sizes indicated next to the nodes. Node 0 is the depot. You have to find the optimal set of vehicle routes that start at the depot and return to the depot, and that serve all the customers. Each vehicle has a capacity of 100 units. A customer may be served by more than one vehicle (split deliveries are allowed). You have to use the minimum number of vehicles.

1. Determine the minimum number of vehicles needed.
   Answer:

$$\text{Minimum number of vehicles needed} = \left\lceil \frac{65+45+25+35+70+50}{100} \right\rceil = \left\lceil \frac{290}{100} \right\rceil = 3$$

2. Write down a complete integer linear programming formulation for the vehicle routing problem on the network. With "complete" is meant that you should write the objective function and each individual constraint with the given data substituted in the appropriate places.
   Answer: Let $N = \{0,1,2,3,4,5,6\}$ denote the set of nodes. Let $E = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\},\{1,2\},\{1,3\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,6\},\{3,4\},\{3,5\},\{4,5\},\{4,6\},\{5,6\}\}$ denote the set of edges. Let $G = (N,E)$ denote the given undirected network where node 0 is the depot.
   Let c_{ij} denote the cost associated with edge $\{i,j\}$.
   Let d_i denote the demand at $i \in N \setminus \{0\}$.
   Let K denote the number of vehicles needed.
   Decision variables:
   Let $x_{ij}^k = \begin{cases} 1 & \text{if vehicle } k \text{ travels on edge } \{i,j\} \in E \\ 0 & \text{o/w} \end{cases}$
   Let $y_i^k = \begin{cases} 1 & \text{if vehicle } k \text{ visits customer } i \in V \setminus \{0\} \\ 0 & \text{o/w} \end{cases}$
   Let $q_i^k \in [0, d_i]$ denote the amount delivered by vehicle k at customer $i \in N \setminus \{0\}$.
   Then the formulation of the problem is as follows:

$$
\begin{array}{lrcll}
\min & \multicolumn{4}{l}{\sum_{k=1}^{K} \sum_{\{i,j\}\in E} c_{ij} x_{ij}^k} \\
\text{s.t.} & & & & \\
& \sum_{\{j\in N:\{j,i\}\in E\}} x_{ji}^k + \sum_{\{j\in N:\{i,j\}\in E\}} x_{ij}^k & = & 2y_i^k & \forall\, k \in \{1,\dots,K\}, \forall\, i \in N\setminus\{0\} \\
& \sum_{k=1}^{K} q_i^k & = & d_i & \forall\, i \in N\setminus\{0\} \\
& q_i^k & \le & \min\{100, d_i\} y_i^k & \forall\, k \in \{1,\dots,K\}, \forall\, i \in N\setminus\{0\} \\
& \sum_{i\in N\setminus\{0\}} q_i^k & \le & 100 & \forall\, k \in \{1,\dots,K\} \\
& \sum_{\{\{i,j\}\in E:\, i,j\in S\}} x_{ij}^k & \le & |S|-1 & \forall\, k \in \{1,\dots,K\},\ \forall\, S \subset N\setminus\{0\} \ :\ 3 \le |S| \le |N-1| \\
& x_{ij}^k & \in & \{0,1\} & \forall\, \{i,j\} \in E,\ \forall\, k \in \{1,\dots,K\} \\
& y_i^k & \in & \{0,1\} & \forall\, i \in N\setminus\{0\},\ \forall\, k \in \{1,\dots,K\} \\
& q_i^k & \ge & 0 & \forall\, i \in N\setminus\{0\},\ \forall\, k \in \{1,\dots,K\}
\end{array}
$$

With the given data, the formulation is

$$
\begin{aligned}
\min \quad & \{30(x^1_{0,1}+x^2_{0,1}+x^3_{0,1})+27(x^1_{0,2}+x^2_{0,2}+x^3_{0,2})\\
& +33(x^1_{0,3}+x^2_{0,3}+x^3_{0,3})+37(x^1_{0,4}+x^2_{0,4}+x^3_{0,4})\\
& +23(x^1_{0,5}+x^2_{0,5}+x^3_{0,5})+34(x^1_{0,6}+x^2_{0,6}+x^3_{0,6})\\
& +25(x^1_{1,2}+x^2_{1,2}+x^3_{1,2})+35(x^1_{1,3}+x^2_{1,3}+x^3_{1,3})\\
& +31(x^1_{1,5}+x^2_{1,5}+x^3_{1,5})+21(x^1_{1,6}+x^2_{1,6}+x^3_{1,6})\\
& +22(x^1_{2,3}+x^2_{2,3}+x^3_{2,3})+32(x^1_{2,4}+x^2_{2,4}+x^3_{2,4})\\
& +29(x^1_{2,6}+x^2_{2,6}+x^3_{2,6})+24(x^1_{3,4}+x^2_{3,4}+x^3_{3,4})\\
& +36(x^1_{3,5}+x^2_{3,5}+x^3_{3,5})+28(x^1_{4,5}+x^2_{4,5}+x^3_{4,5})\\
& +38(x^1_{4,6}+x^2_{4,6}+x^3_{4,6})+26(x^1_{5,6}+x^2_{5,6}+x^3_{5,6})\}\\
\text{s.t.} \quad & x^1_{01}+x^1_{12}+x^1_{13}+x^1_{15}+x^1_{16} = 2y^1_1\\
& x^2_{01}+x^2_{12}+x^2_{13}+x^2_{15}+x^2_{16} = 2y^2_1\\
& x^3_{01}+x^3_{12}+x^3_{13}+x^3_{15}+x^3_{16} = 2y^3_1\\
& x^1_{02}+x^1_{12}+x^1_{23}+x^1_{24}+x^1_{26} = 2y^1_2\\
& x^2_{02}+x^2_{12}+x^2_{23}+x^2_{24}+x^2_{26} = 2y^2_2\\
& x^3_{02}+x^3_{12}+x^3_{23}+x^3_{24}+x^3_{26} = 2y^3_2\\
& x^1_{03}+x^1_{13}+x^1_{23}+x^1_{34}+x^1_{35} = 2y^1_3\\
& x^2_{03}+x^2_{13}+x^2_{23}+x^2_{34}+x^2_{35} = 2y^2_3\\
& x^3_{03}+x^3_{13}+x^3_{23}+x^3_{34}+x^3_{35} = 2y^3_3\\
& x^1_{04}+x^1_{24}+x^1_{34}+x^1_{45}+x^1_{46} = 2y^1_4\\
& x^2_{04}+x^2_{24}+x^2_{34}+x^2_{45}+x^2_{46} = 2y^2_4\\
& x^3_{04}+x^3_{24}+x^3_{34}+x^3_{45}+x^3_{46} = 2y^3_4\\
& x^1_{05}+x^1_{15}+x^1_{35}+x^1_{45}+x^1_{56} = 2y^1_5\\
& x^2_{05}+x^2_{15}+x^2_{35}+x^2_{45}+x^2_{56} = 2y^2_5\\
& x^3_{05}+x^3_{15}+x^3_{35}+x^3_{45}+x^3_{56} = 2y^3_5\\
& x^1_{06}+x^1_{16}+x^1_{26}+x^1_{46}+x^1_{56} = 2y^1_6\\
& x^2_{06}+x^2_{16}+x^2_{26}+x^2_{46}+x^2_{56} = 2y^2_6\\
& x^3_{06}+x^3_{16}+x^3_{26}+x^3_{46}+x^3_{56} = 2y^3_6\\
& q^1_1+q^2_1+q^3_1 = 65\\
& q^1_2+q^2_2+q^3_2 = 45\\
& q^1_3+q^2_3+q^3_3 = 25\\
& q^1_4+q^2_4+q^3_4 = 35\\
& q^1_5+q^2_5+q^3_5 = 70\\
& q^1_6+q^2_6+q^3_6 = 50\\
& q^1_1 \le 65y^1_1\\
& q^2_1 \le 65y^2_1\\
& q^3_1 \le 65y^3_1\\
& q^1_2 \le 45y^1_2\\
& q^2_2 \le 45y^2_2\\
& q^3_2 \le 45y^3_2
\end{aligned}
$$

$$
\begin{aligned}
q_3^1 &\leq 25y_3^1 \\
q_3^2 &\leq 25y_3^2 \\
q_3^3 &\leq 25y_3^3 \\
q_4^1 &\leq 35y_4^1 \\
q_4^2 &\leq 35y_4^2 \\
q_4^3 &\leq 35y_4^3 \\
q_5^1 &\leq 70y_5^1 \\
q_5^2 &\leq 70y_5^2 \\
q_5^3 &\leq 70y_5^3 \\
q_6^1 &\leq 50y_6^1 \\
q_6^2 &\leq 50y_6^2 \\
q_6^3 &\leq 50y_6^3 \\
q_1^1 + q_2^1 + q_3^1 + q_4^1 + q_5^1 + q_6^1 &\leq 100 \\
q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 &\leq 100 \\
q_1^3 + q_2^3 + q_3^3 + q_4^3 + q_5^3 + q_6^3 &\leq 100 \\
x_{12}^1 + x_{13}^1 + x_{23}^1 &\leq 2 && S = \{1,2,3\}, k = 1 \\
x_{12}^2 + x_{13}^2 + x_{23}^2 &\leq 2 && S = \{1,2,3\}, k = 2 \\
x_{12}^3 + x_{13}^3 + x_{23}^3 &\leq 2 && S = \{1,2,3\}, k = 3 \\
x_{12}^1 + x_{16}^1 + x_{26}^1 &\leq 2 && S = \{1,2,6\}, k = 1 \\
x_{12}^2 + x_{16}^2 + x_{26}^2 &\leq 2 && S = \{1,2,6\}, k = 2 \\
x_{12}^3 + x_{16}^3 + x_{26}^3 &\leq 2 && S = \{1,2,6\}, k = 3 \\
x_{13}^1 + x_{15}^1 + x_{35}^1 &\leq 2 && S = \{1,3,5\}, k = 1 \\
x_{13}^2 + x_{15}^2 + x_{35}^2 &\leq 2 && S = \{1,3,5\}, k = 2 \\
x_{13}^3 + x_{15}^3 + x_{35}^3 &\leq 2 && S = \{1,3,5\}, k = 3 \\
x_{15}^1 + x_{16}^1 + x_{56}^1 &\leq 2 && S = \{1,5,6\}, k = 1 \\
x_{15}^2 + x_{16}^2 + x_{56}^2 &\leq 2 && S = \{1,5,6\}, k = 2 \\
x_{15}^3 + x_{16}^3 + x_{56}^3 &\leq 2 && S = \{1,5,6\}, k = 3 \\
x_{23}^1 + x_{24}^1 + x_{34}^1 &\leq 2 && S = \{2,3,4\}, k = 1 \\
x_{23}^2 + x_{24}^2 + x_{34}^2 &\leq 2 && S = \{2,3,4\}, k = 2 \\
x_{23}^3 + x_{24}^3 + x_{34}^3 &\leq 2 && S = \{2,3,4\}, k = 3 \\
x_{24}^1 + x_{26}^1 + x_{46}^1 &\leq 2 && S = \{2,4,6\}, k = 1 \\
x_{24}^2 + x_{26}^2 + x_{46}^2 &\leq 2 && S = \{2,4,6\}, k = 2 \\
x_{24}^3 + x_{26}^3 + x_{46}^3 &\leq 2 && S = \{2,4,6\}, k = 3 \\
x_{34}^1 + x_{35}^1 + x_{45}^1 &\leq 2 && S = \{3,4,5\}, k = 1 \\
x_{34}^2 + x_{35}^2 + x_{45}^2 &\leq 2 && S = \{3,4,5\}, k = 2 \\
x_{34}^3 + x_{35}^3 + x_{45}^3 &\leq 2 && S = \{3,4,5\}, k = 3 \\
x_{45}^1 + x_{46}^1 + x_{56}^1 &\leq 2 && S = \{4,5,6\}, k = 1 \\
x_{45}^2 + x_{46}^2 + x_{56}^2 &\leq 2 && S = \{4,5,6\}, k = 2 \\
x_{45}^3 + x_{46}^3 + x_{56}^3 &\leq 2 && S = \{4,5,6\}, k = 3 \\
x_{12}^1 + x_{13}^1 + x_{23}^1 + x_{24}^1 + x_{34}^1 &\leq 3 && S = \{1,2,3,4\}, k = 1 \\
x_{12}^2 + x_{13}^2 + x_{23}^2 + x_{24}^2 + x_{34}^2 &\leq 3 && S = \{1,2,3,4\}, k = 2
\end{aligned}
$$

$$
\begin{aligned}
x^3_{12}+x^3_{13}+x^3_{23}+x^3_{24}+x^3_{34} &\leq 3 && S=\{1,2,3,4\}, k=3\\
x^1_{12}+x^1_{13}+x^1_{15}+x^1_{23}+x^1_{35} &\leq 3 && S=\{1,2,3,5\}, k=1\\
x^2_{12}+x^2_{13}+x^2_{15}+x^2_{23}+x^2_{35} &\leq 3 && S=\{1,2,3,5\}, k=2\\
x^3_{12}+x^3_{13}+x^3_{15}+x^3_{23}+x^3_{35} &\leq 3 && S=\{1,2,3,5\}, k=3\\
x^1_{12}+x^1_{13}+x^1_{16}+x^1_{23}+x^1_{26} &\leq 3 && S=\{1,2,3,6\}, k=1\\
x^2_{12}+x^2_{13}+x^2_{16}+x^2_{23}+x^2_{26} &\leq 3 && S=\{1,2,3,6\}, k=2\\
x^3_{12}+x^3_{13}+x^3_{16}+x^3_{23}+x^3_{26} &\leq 3 && S=\{1,2,3,6\}, k=3\\
x^1_{13}+x^1_{15}+x^1_{34}+x^1_{35}+x^1_{45} &\leq 3 && S=\{1,3,4,5\}, k=1\\
x^2_{13}+x^2_{15}+x^2_{34}+x^2_{35}+x^2_{45} &\leq 3 && S=\{1,3,4,5\}, k=2\\
x^3_{13}+x^3_{15}+x^3_{34}+x^3_{35}+x^3_{45} &\leq 3 && S=\{1,3,4,5\}, k=3\\
x^1_{13}+x^1_{16}+x^1_{34}+x^1_{46} &\leq 3 && S=\{1,3,4,6\}, k=1\\
x^2_{13}+x^2_{16}+x^2_{34}+x^2_{46} &\leq 3 && S=\{1,3,4,6\}, k=2\\
x^3_{13}+x^3_{16}+x^3_{34}+x^3_{46} &\leq 3 && S=\{1,3,4,6\}, k=3\\
x^1_{13}+x^1_{15}+x^1_{16}+x^1_{35}+x^1_{56} &\leq 3 && S=\{1,3,5,6\}, k=1\\
x^2_{13}+x^2_{15}+x^2_{16}+x^2_{35}+x^2_{56} &\leq 3 && S=\{1,3,5,6\}, k=2\\
x^3_{13}+x^3_{15}+x^3_{16}+x^3_{35}+x^3_{56} &\leq 3 && S=\{1,3,5,6\}, k=3\\
x^1_{15}+x^1_{16}+x^1_{45}+x^1_{46}+x^1_{56} &\leq 3 && S=\{1,4,5,6\}, k=1\\
x^2_{15}+x^2_{16}+x^2_{45}+x^2_{46}+x^2_{56} &\leq 3 && S=\{1,4,5,6\}, k=2\\
x^3_{15}+x^3_{16}+x^3_{45}+x^3_{46}+x^3_{56} &\leq 3 && S=\{1,4,5,6\}, k=3\\
x^1_{23}+x^1_{24}+x^1_{34}+x^1_{35}+x^1_{45} &\leq 3 && S=\{2,3,4,5\}, k=1\\
x^2_{23}+x^2_{24}+x^2_{34}+x^2_{35}+x^2_{45} &\leq 3 && S=\{2,3,4,5\}, k=2\\
x^3_{23}+x^3_{24}+x^3_{34}+x^3_{35}+x^3_{45} &\leq 3 && S=\{2,3,4,5\}, k=3\\
x^1_{23}+x^1_{24}+x^1_{26}+x^1_{34}+x^1_{46} &\leq 3 && S=\{2,3,4,6\}, k=1\\
x^2_{23}+x^2_{24}+x^2_{26}+x^2_{34}+x^2_{46} &\leq 3 && S=\{2,3,4,6\}, k=2\\
x^3_{23}+x^3_{24}+x^3_{26}+x^3_{34}+x^3_{46} &\leq 3 && S=\{2,3,4,6\}, k=3\\
x^1_{23}+x^1_{26}+x^1_{35}+x^1_{56} &\leq 3 && S=\{2,3,5,6\}, k=1\\
x^2_{23}+x^2_{26}+x^2_{35}+x^2_{56} &\leq 3 && S=\{2,3,5,6\}, k=2\\
x^3_{23}+x^3_{26}+x^3_{35}+x^3_{56} &\leq 3 && S=\{2,3,5,6\}, k=3\\
x^1_{24}+x^1_{26}+x^1_{45}+x^1_{46}+x^1_{56} &\leq 3 && S=\{2,4,5,6\}, k=1\\
x^2_{24}+x^2_{26}+x^2_{45}+x^2_{46}+x^2_{56} &\leq 3 && S=\{2,4,5,6\}, k=2\\
x^3_{24}+x^3_{26}+x^3_{45}+x^3_{46}+x^3_{56} &\leq 3 && S=\{2,4,5,6\}, k=3\\
x^1_{34}+x^1_{35}+x^1_{45}+x^1_{46}+x^1_{56} &\leq 3 && S=\{3,4,5,6\}, k=1\\
x^2_{34}+x^2_{35}+x^2_{45}+x^2_{46}+x^2_{56} &\leq 3 && S=\{3,4,5,6\}, k=2\\
x^3_{34}+x^3_{35}+x^3_{45}+x^3_{46}+x^3_{56} &\leq 3 && S=\{3,4,5,6\}, k=3\\
x^1_{12}+x^1_{13}+x^1_{15}+x^1_{23}+x^1_{24}+x^1_{34}+x^1_{35}+x^1_{45} &\leq 4 && S=\{1,2,3,4,5\}, k=1\\
x^2_{12}+x^2_{13}+x^2_{15}+x^2_{23}+x^2_{24}+x^2_{34}+x^2_{35}+x^2_{45} &\leq 4 && S=\{1,2,3,4,5\}, k=2\\
x^3_{12}+x^3_{13}+x^3_{15}+x^3_{23}+x^3_{24}+x^3_{34}+x^3_{35}+x^3_{45} &\leq 4 && S=\{1,2,3,4,5\}, k=3\\
x^1_{12}+x^1_{13}+x^1_{16}+x^1_{23}+x^1_{24}+x^1_{26}+x^1_{34}+x^1_{46} &\leq 4 && S=\{1,2,3,4,6\}, k=1\\
x^2_{12}+x^2_{13}+x^2_{16}+x^2_{23}+x^2_{24}+x^2_{26}+x^2_{34}+x^2_{46} &\leq 4 && S=\{1,2,3,4,6\}, k=2\\
x^3_{12}+x^3_{13}+x^3_{16}+x^3_{23}+x^3_{24}+x^3_{26}+x^3_{34}+x^3_{46} &\leq 4 && S=\{1,2,3,4,6\}, k=3\\
x^1_{12}+x^1_{13}+x^1_{15}+x^1_{16}+x^1_{23}+x^1_{26}+x^1_{35}+x^1_{56} &\leq 4 && S=\{1,2,3,5,6\}, k=1
\end{aligned}
$$

$$
\begin{array}{rcll}
x_{12}^2 + x_{13}^2 + x_{15}^2 + x_{16}^2 + x_{23}^2 + x_{26}^2 + x_{35}^2 + x_{56}^2 & \leq & 4 & S = \{1,2,3,5,6\}, k = 2 \\
x_{12}^3 + x_{13}^3 + x_{15}^3 + x_{16}^3 + x_{23}^3 + x_{26}^3 + x_{35}^3 + x_{56}^3 & \leq & 4 & S = \{1,2,3,5,6\}, k = 3 \\
x_{12}^1 + x_{15}^1 + x_{16}^1 + x_{24}^1 + x_{26}^1 + x_{45}^1 + x_{46}^1 + x_{56}^1 & \leq & 4 & S = \{1,2,4,5,6\}, k = 1 \\
x_{12}^2 + x_{15}^2 + x_{16}^2 + x_{24}^2 + x_{26}^2 + x_{45}^2 + x_{46}^2 + x_{56}^2 & \leq & 4 & S = \{1,2,4,5,6\}, k = 2 \\
x_{12}^3 + x_{15}^3 + x_{16}^3 + x_{24}^3 + x_{26}^3 + x_{45}^3 + x_{46}^3 + x_{56}^3 & \leq & 4 & S = \{1,2,4,5,6\}, k = 3 \\
x_{13}^1 + x_{15}^1 + x_{16}^1 + x_{34}^1 + x_{35}^1 + x_{45}^1 + x_{46}^1 + x_{56}^1 & \leq & 4 & S = \{1,3,4,5,6\}, k = 1 \\
x_{13}^2 + x_{15}^2 + x_{16}^2 + x_{34}^2 + x_{35}^2 + x_{45}^2 + x_{46}^2 + x_{56}^2 & \leq & 4 & S = \{1,3,4,5,6\}, k = 2 \\
x_{13}^3 + x_{15}^3 + x_{16}^3 + x_{34}^3 + x_{35}^3 + x_{45}^3 + x_{46}^3 + x_{56}^3 & \leq & 4 & S = \{1,3,4,5,6\}, k = 3 \\
x_{23}^1 + x_{24}^1 + x_{26}^1 + x_{34}^1 + x_{35}^1 + x_{45}^1 + x_{46}^1 + x_{56}^1 & \leq & 4 & S = \{2,3,4,5,6\}, k = 1 \\
x_{23}^2 + x_{24}^2 + x_{26}^2 + x_{34}^2 + x_{35}^2 + x_{45}^2 + x_{46}^2 + x_{56}^2 & \leq & 4 & S = \{2,3,4,5,6\}, k = 2 \\
x_{23}^3 + x_{24}^3 + x_{26}^3 + x_{34}^3 + x_{35}^3 + x_{45}^3 + x_{46}^3 + x_{56}^3 & \leq & 4 & S = \{2,3,4,5,6\}, k = 3 \\
x_{12}^1 + x_{13}^1 + x_{15}^1 + x_{16}^1 + x_{23}^1 + x_{24}^1 + x_{26}^1 + x_{34}^1 + x_{35}^1 + x_{45}^1 + x_{46}^1 + x_{56}^1 & \leq & 5 & S = \{1,2,3,4,5,6\}, k = 1 \\
x_{12}^2 + x_{13}^2 + x_{15}^2 + x_{16}^2 + x_{23}^2 + x_{24}^2 + x_{26}^2 + x_{34}^2 + x_{35}^2 + x_{45}^2 + x_{46}^2 + x_{56}^2 & \leq & 5 & S = \{1,2,3,4,5,6\}, k = 2 \\
x_{12}^3 + x_{13}^3 + x_{15}^3 + x_{16}^3 + x_{23}^3 + x_{24}^3 + x_{26}^3 + x_{34}^3 + x_{35}^3 + x_{45}^3 + x_{46}^3 + x_{56}^3 & \leq & 5 & S = \{1,2,3,4,5,6\}, k = 3 \\
x_{0,1}^1, x_{0,1}^2, x_{0,1}^3, x_{0,2}^1, x_{0,2}^2, x_{0,2}^3, x_{0,3}^1, x_{0,3}^2, x_{0,3}^3, x_{0,4}^1, x_{0,4}^2, x_{0,4}^3, x_{0,5}^1, x_{0,5}^2, x_{0,5}^3 & \in & \{0,1\} & \\
x_{0,6}^1, x_{0,6}^2, x_{0,6}^3, x_{1,2}^1, x_{1,2}^2, x_{1,2}^3, x_{1,3}^1, x_{1,3}^2, x_{1,3}^3, x_{1,5}^1, x_{1,5}^2, x_{1,5}^3, x_{1,6}^1, x_{1,6}^2, x_{1,6}^3 & \in & \{0,1\} & \\
x_{2,3}^1, x_{2,3}^2, x_{2,3}^3, x_{2,4}^1, x_{2,4}^2, x_{2,4}^3, x_{2,6}^1, x_{2,6}^2, x_{2,6}^3, x_{3,4}^1, x_{3,4}^2, x_{3,4}^3, x_{3,5}^1, x_{3,5}^2, x_{3,5}^3 & \in & \{0,1\} & \\
x_{4,5}^1, x_{4,5}^2, x_{4,5}^3, x_{4,6}^1, x_{4,6}^2, x_{4,6}^3, x_{5,6}^1, x_{5,6}^2, x_{5,6}^3 & \in & \{0,1\} & \\
y_1^1, y_1^2, y_1^3, y_2^1, y_2^2, y_2^3, y_3^1, y_3^2, y_3^3 & \in & \{0,1\} & \\
y_4^1, y_4^2, y_4^3, y_5^1, y_5^2, y_5^3, y_6^1, y_6^2, y_6^3 & \in & \{0,1\} & \\
q_1^1, q_1^2, q_1^3, q_2^1, q_2^2, q_2^3, q_3^1, q_3^2, q_3^3 & \geq & 0 & \\
q_4^1, q_4^2, q_4^3, q_5^1, q_5^2, q_5^3, q_6^1, q_6^2, q_6^3 & \geq & 0 &
\end{array}
$$

3. Find an optimal solution for the vehicle routing problem, using a solver of your choice. Clearly present your solution and optimal objective value.
Answer: See Figure 1:

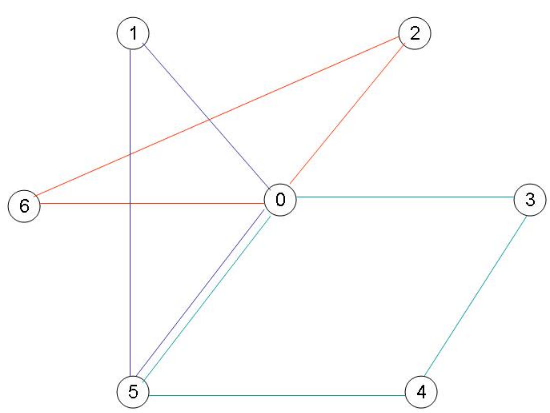

Figure 1: Total Cost = 282

Question 2 (25 points)

Below is a network, with costs indicated on the arcs, and load sizes indicated next to the nodes. Node 0 is the depot. The nodes indicated with circles are origin nodes, where loads have to be picked up. The nodes indicated with squares are destination nodes, where loads have to be delivered. A vehicle can do both deliveries and pickups on a single route, but it has to do all deliveries on the route before any pickups on the route. You have to find the optimal set of vehicle routes that start at the depot and return to the depot, and that serve all the customers. Each vehicle has a capacity of 100 units. Each customer must be served by exactly one vehicle (split deliveries are not allowed).

1. Write down a complete integer linear programming formulation for the vehicle routing problem on the network. With "complete" is meant that you should write the objective function and each individual constraint with the given data substituted in the appropriate places.

   Answer: Let $O = \{2, 4, 6, 8\}$ denote the set of origin (pickup) nodes, and let $D = \{1, 3, 5, 7\}$ denote the set of destination (delivery) nodes. Let $N = \{0\} \cup O \cup D$ denote the set of all the nodes. Convert the given undirected edges into directed arcs in the following way: For each undirected edge $\{0, i\}, i \in O \cup D$, create two directed arcs, $(0, i)$ and $(i, 0)$, with the same cost as edge $\{0, i\}$. For each undirected edge $\{i, j\}, i, j \in O$, create two directed arcs, (i, j) and (j, i), with the same cost as edge $\{i, j\}$. For each undirected edge $\{i, j\}, i, j \in D$, create two directed arcs, (i, j) and (j, i), with the same cost as edge $\{i, j\}$. For each undirected edge $\{i, j\}$ or $\{j, i\}$, $i \in O, j \in D$, create only one directed arc, (j, i), (because deliveries must precede pickups on a route) with the same cost as edge $\{i, j\}$ or $\{j, i\}$. The resulting set of arcs is $A = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (1, 0), (1, 2), (1, 3), (1, 7), (1, 8), (2, 0), (2, 4), (2, 8), (3, 0), (3, 1), (3, 2), (3, 4), (3, 5), (4, 0), (4, 2), (4, 6), (5, 0), (5, 2), (5, 3), (5, 4), (5, 6), (5, 7), (5, 8), (6, 0), (6, 4), (6, 8), (7, 0), (7, 1), (7, 2), (7, 4), (7, 5), (7, 6), (7, 8), (8, 0), (8, 2), (8, 6)\}$. Let $G = (N, A)$ denote the resulting directed network where node 0 is the depot.

   Let c_{ij} denote the cost associated with arc (i, j).

   Let d_i denote the quantity to be delivered or picked up at $i \in O \cup D$.

   Note that a subtour cannot include both origin nodes and destination nodes, because there are no arcs from origin nodes to destination nodes. For any subset S of origin nodes, $S \subset O$, let $\alpha(S)$ denote the minimum number of vehicles needed to serve all customers in S. Similarly, for any subset S of destination nodes, $S \subset D$, let $\alpha(S)$ denote the minimum number of vehicles needed to serve all customers in S.

   Assume that if a vehicle visits a node, then it serves that node.

   Decision variables:

   Let $x_{ij} = \begin{cases} 1 & \text{if a vehicle travels on arc } (i,j) \in A \\ 0 & \text{o/w} \end{cases}$

   Then the formulation of the problem is as follows:

$$\begin{array}{llll} \min & \sum_{(i,j)\in A} c_{ij} x_{ij} & & \\ \text{s.t.} & & & \\ & \sum_{\{j \in N : (j,i) \in A\}} x_{ji} & = 1 & \forall\, i \in O \cup D \\ & \sum_{\{j \in N : (i,j) \in A\}} x_{ij} & = 1 & \forall\, i \in O \cup D \\ & \sum_{\{(i,j) \in A : , i \in S, j \notin S\}} x_{ij} & \geq \alpha(S) & \forall\, S \subset O \,:\, 2 \leq |S| \leq |O|, \quad \forall\, S \subset D \,:\, 2 \leq |S| \leq |D| \\ & x_{ij} & \in \{0, 1\} & \forall\, (i,j) \in A \end{array}$$

   With the given data, the formulation is

$$\begin{aligned} \min \quad & \{45x_{0,1} + 38x_{0,2} + 44x_{0,3} + 43x_{0,4} + 46x_{0,5} + 39x_{0,6} + 41x_{0,7} + 37x_{0,8} \\ & +45x_{1,0} + 23x_{1,2} + 30x_{1,3} + 32x_{1,7} + 31x_{1,8} + 38x_{2,0} + 35x_{2,4} + 36x_{2,8} + 44x_{3,0} \\ & +30x_{3,1} + 27x_{3,2} + 24x_{3,4} + 20x_{3,5} + 43x_{4,0} + 35x_{4,2} + 42x_{4,6} + 46x_{5,0} + 20x_{5,3} \end{aligned}$$

$$
\begin{aligned}
&+34x_{5,4}+25x_{5,6}+33x_{5,7}+40x_{5,8}+39x_{6,0}+42x_{6,4}+28x_{6,8}+41x_{7,0}+32x_{7,1}\\
&+29x_{7,2}+26x_{7,4}+33x_{7,5}+22x_{7,6}+21x_{7,8}+37x_{8,0}+36x_{8,2}+28x_{8,6}\}
\end{aligned}
$$

$$
\begin{array}{rcll}
\text{s.t.} \quad x_{0,1}+x_{3,1}+x_{7,1} & = & 1 & \\
x_{0,2}+x_{1,2}+x_{3,2}+x_{4,2}+x_{7,2}+x_{8,2} & = & 1 & \\
x_{0,3}+x_{1,3}+x_{5,3} & = & 1 & \\
x_{0,4}+x_{2,4}+x_{3,4}+x_{5,4}+x_{6,4}+x_{7,4} & = & 1 & \\
x_{0,5}+x_{3,5}+x_{7,5} & = & 1 & \\
x_{0,6}+x_{4,6}+x_{5,6}+x_{7,6}+x_{8,6} & = & 1 & \\
x_{0,7}+x_{1,7}+x_{5,7} & = & 1 & \\
x_{0,8}+x_{1,8}+x_{2,8}+x_{5,8}+x_{6,8}+x_{7,8} & = & 1 & \\
x_{1,0}+x_{1,2}+x_{1,3}+x_{1,7}+x_{1,8} & = & 1 & \\
x_{2,0}+x_{2,4}+x_{2,8} & = & 1 & \\
x_{3,0}+x_{3,1}+x_{3,2}+x_{3,4}+x_{3,5} & = & 1 & \\
x_{4,0}+x_{4,2}+x_{4,6} & = & 1 & \\
x_{5,0}+x_{5,2}+x_{5,3}+x_{5,4}+x_{5,6}+x_{5,7}+x_{5,8} & = & 1 & \\
x_{6,0}+x_{6,4}+x_{6,8} & = & 1 & \\
x_{7,0}+x_{7,1}+x_{7,2}+x_{7,4}+x_{7,5}+x_{7,6}+x_{7,8} & = & 1 & \\
x_{8,0}+x_{8,2}+x_{8,6} & = & 1 & \\
x_{1,0}+x_{1,2}+x_{1,7}+x_{1,8}+x_{3,0}+x_{3,2}+x_{3,4}+x_{3,5} & \geq & 1 & S=\{1,3\}\\
x_{1,0}+x_{1,2}+x_{1,3}+x_{1,8}+x_{7,0}+x_{7,2}+x_{7,4}+x_{7,5}+x_{7,6}+x_{7,8} & \geq & 1 & S=\{1,7\}\\
x_{5,0}+x_{5,3}+x_{5,4}+x_{5,6}+x_{5,8}+x_{7,0}+x_{7,1}+x_{7,2}+x_{7,4}+x_{7,6}+x_{7,8} & \geq & 1 & S=\{5,7\}\\
x_{1,0}+x_{1,2}+x_{1,7}+x_{1,8}+x_{3,0}+x_{3,2}+x_{3,4} & & & \\
+x_{5,0}+x_{5,4}+x_{5,6}+x_{5,7}+x_{5,8} & \geq & 2 & S=\{1,3,5\}\\
x_{1,0}+x_{1,2}+x_{1,8}+x_{3,0}+x_{3,2}+x_{3,4}+x_{3,5} & & & \\
+x_{7,0}+x_{7,2}+x_{7,4}+x_{7,5}+x_{7,6}+x_{7,8} & \geq & 2 & S=\{1,3,7\}\\
x_{1,0}+x_{1,2}+x_{1,3}+x_{1,8}+x_{5,0}+x_{5,3}+x_{5,4}+x_{5,6}+x_{5,8} & & & \\
+x_{7,0}+x_{7,1}+x_{7,2}+x_{7,4}+x_{7,6}+x_{7,8} & \geq & 2 & S=\{1,5,7\}\\
x_{3,0}+x_{3,1}+x_{3,2}+x_{3,4}+x_{5,0}+x_{5,4}+x_{5,6}+x_{5,8} & & & \\
+x_{7,0}+x_{7,1}+x_{7,2}+x_{7,4}+x_{7,6}+x_{7,8} & \geq & 2 & S=\{3,5,7\}\\
x_{1,0}+x_{1,2}+x_{1,8}+x_{3,0}+x_{3,2}+x_{3,4}+x_{5,0}+x_{5,4}+x_{5,6}+x_{5,8} & & & \\
+x_{7,0}+x_{7,2}+x_{7,4}+x_{7,6}+x_{7,8} & \geq & 2 & S=\{1,3,5,7\}\\
x_{2,0}+x_{2,8}+x_{4,0}+x_{4,6} & \geq & 1 & S=\{2,4\}\\
x_{2,0}+x_{2,4}+x_{8,0}+x_{8,6} & \geq & 1 & S=\{2,8\}\\
x_{4,0}+x_{4,2}+x_{6,0}+x_{6,8} & \geq & 1 & S=\{4,6\}\\
x_{6,0}+x_{6,4}+x_{8,0}+x_{8,2} & \geq & 2 & S=\{6,8\}\\
x_{2,0}+x_{2,8}+x_{4,0}+x_{6,0}+x_{6,8} & \geq & 2 & S=\{2,4,6\}\\
x_{2,0}+x_{4,0}+x_{4,6}+x_{8,0}+x_{8,6} & \geq & 2 & S=\{2,4,8\}\\
x_{2,0}+x_{2,4}+x_{6,0}+x_{6,4}+x_{8,0} & \geq & 2 & S=\{2,6,8\}\\
x_{4,0}+x_{4,2}+x_{6,0}+x_{8,0}+x_{8,2} & \geq & 2 & S=\{4,6,8\}\\
x_{2,0}+x_{4,0}+x_{6,0}+x_{8,0} & \geq & 2 & S=\{2,4,6,8\}\\
x_{0,1},x_{0,2},x_{0,3},x_{0,4},x_{0,5},x_{0,6},x_{0,7},x_{0,8},x_{1,0},x_{1,2},x_{1,3} & \in & \{0,1\} &
\end{array}
$$

$$x_{1,7}, x_{1,8}, x_{2,0}, x_{2,4}, x_{2,8}, x_{3,0}, x_{3,1}, x_{3,2}, x_{3,4}, x_{4,0}, x_{4,2} \in \{0, 1\}$$
$$x_{4,6}, x_{5,0}, x_{5,2}, x_{5,4}, x_{5,6}, x_{5,7}, x_{5,8}, x_{6,0}, x_{6,4}, x_{6,8} \in \{0, 1\}$$
$$x_{7,0}, x_{7,1}, x_{7,2}, x_{7,4}, x_{7,5}, x_{7,6}, x_{7,8}, x_{8,0}, x_{8,2}, x_{8,6} \in \{0, 1\}$$

2. Find an optimal solution for the vehicle routing problem, using a solver of your choice. Clearly present your solution and optimal objective value.
Answer: See Figure 2:

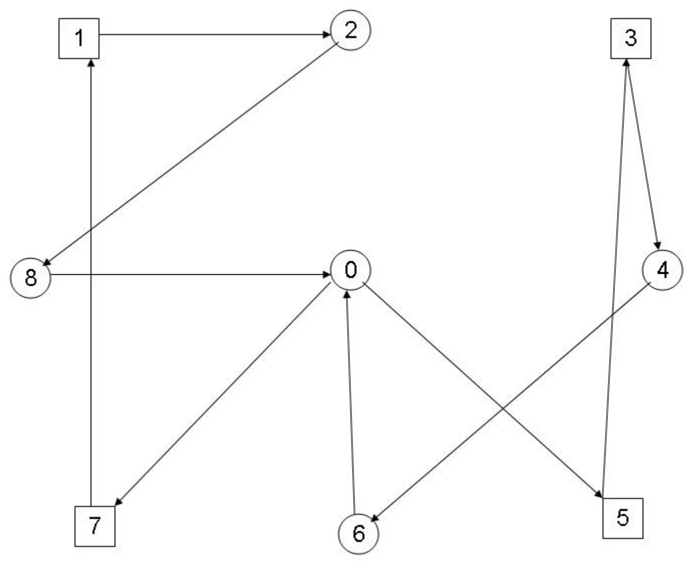

Figure 2: Total Cost = 340